Four Derivations of the Black-Scholes Formula
by Fabrice Douglas Rouah
www.FRouah.com
www.Volopta.com

In this note we derive in four separate ways the well-known result of Black and Scholes that under certain assumptions the time-
t price $C(S_t, K, T)$ of a European call option with strike price $K$ and maturity $\tau = T - t$ on a non-dividend stock with spot price $S_t$ and a constant volatility $\sigma$ when the rate of interest is a constant $r$ can be expressed as

$$C(S_t, K, T) = S_t \Phi(d_1) - e^{-\tau r} K \Phi(d_2)$$  \hspace{1cm} (1)

where

$$d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{\sigma^2}{2}) \tau}{\sigma \sqrt{\tau}}$$

and $d_2 = d_1 - \sigma \sqrt{\tau}$, and where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2} dt$ is the standard normal cdf. We show four ways in which Equation (1) can be derived.

1. By straightforward integration.
2. By applying the Feynman-Kac theorem.
3. By transforming the Black Scholes PDE into the heat equation, for which a solution is known. This is the original approach adopted by Black and Scholes [1].
4. Through the Capital Asset Pricing Model (CAPM).

Free code for the Black-Scholes model can be found at www.Volopta.com.

1 Black-Scholes Economy

There are two assets: a risky stock $S$ and riskless bond $B$. These assets are driven by the SDEs

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$  \hspace{1cm} (2)
$$dB_t = r_t B_t dt$$

The time zero value of the bond is $B_0 = 1$ and that of the stock is $S_0$. The model is valid under certain market assumptions that are described in John
Hull’s book [3]. By Itô’s Lemma the value $V_t$ of a derivative written on the stock follows the diffusion

$$
\frac{dV_t}{dt} = \frac{\partial V}{\partial t} \, dt + \frac{\partial V}{\partial S} \, dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \, (dS)^2
$$

$$
= \left( \frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) \, dt + \left( \sigma S_t \frac{\partial V}{\partial S} \right) \, dW_t.
$$

2 The Lognormal Distribution

2.1 The Lognormal PDF and CDF

In this Note we make extensive use of the fact that if a random variable $Y \in \mathbb{R}$ follows the normal distribution with mean $\mu$ and variance $\sigma^2$, then $X = e^Y$ follows the lognormal distribution with mean

$$
E[X] = e^{\mu + \frac{1}{2} \sigma^2}
$$

and variance

$$
Var[X] = (e^{\sigma^2} - 1) \, e^{2\mu + \sigma^2}.
$$

The pdf for $X$ is

$$
dF_X(x) = \frac{1}{\sigma x \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \ln x - \mu \right)^2 \right)
$$

and the cdf is

$$
F_X(x) = \Phi \left( \frac{\ln x - \mu}{\sigma} \right)
$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2} \, dt$ is the standard normal cdf.

2.2 The Lognormal Conditional Expected Value

The expected value of $X$ conditional on $X > x$ is $L_X(K) = E[X \mid X > x]$. For the lognormal distribution this is, using Equation (6)

$$
L_X(K) = \int_K^\infty \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{\ln x - \mu}{\sigma} \right)^2} \, dx.
$$

Make the change of variable $y = \ln x$ so that $x = e^y$, $dx = e^y \, dy$ and the Jacobian is $e^y$. Hence we have

$$
L_X(K) = \int_{\ln K}^{\infty} \frac{e^y}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{y - \mu}{\sigma} \right)^2} \, dy.
$$
Combining terms and completing the square, the exponent is
\[-\frac{1}{2\sigma^2} (y^2 - 2y\mu + \mu^2 - 2\sigma^2 y) = -\frac{1}{2\sigma^2} (y - (\mu + \sigma^2))^2 + \mu + \frac{1}{2}\sigma^2.\]

Equation (8) becomes
\[L_X(K) = \exp (\mu + \frac{1}{2}\sigma^2) \frac{1}{\sigma} \int_{\ln K}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} \left( \frac{y - (\mu + \sigma^2)}{\sigma} \right)^2 \right) dy. \quad (9)\]

Consider the random variable $X$ with pdf $f_X(x)$ and cdf $F_X(x)$, and the scale-location transformation $Y = \sigma X + \mu$. It is easy to show that the Jacobian is $\frac{1}{\sigma}$, that the pdf for $Y$ is $f_Y(y) = \frac{1}{\sigma} f_X \left( \frac{y - \mu}{\sigma} \right)$ and that the cdf is $F_Y(y) = F_X \left( \frac{y - \mu}{\sigma} \right)$.

Hence, the integral in Equation (9) involves the scale-location transformation of the standard normal cdf. Using the fact that $\Phi(-x) = 1 - \Phi(x)$ this implies that
\[L_X(K) = \exp \left( \mu + \frac{\sigma^2}{2} \right) \Phi \left( -\frac{\ln K + \mu + \sigma^2}{\sigma} \right). \quad (10)\]

See Hogg and Klugman [2].

3 Solving the SDEs

3.1 Stock Price

Apply Itô’s Lemma to the function $\ln S_t$ where $S_t$ is driven by the diffusion in Equation (2). Then $\ln S_t$ follows the SDE
\[d\ln S_t = (\mu - \frac{1}{2}\sigma^2) dt + \sigma dW_t. \quad (11)\]

Integrating from 0 to $t$, we have
\[\int_0^t d\ln S_u = \int_0^t (\mu - \frac{1}{2}\sigma^2) du + \sigma \int_0^t dW_u\]
so that
\[\ln S_t - \ln S_0 = (\mu - \frac{1}{2}\sigma^2) t + \sigma W_t\

since $W_0 = 0$. Hence the solution to the SDE is
\[S_t = S_0 \exp \left( (\mu - \frac{1}{2}\sigma^2) t + \sigma W_t \right). \quad (12)\]

Since $W_t$ is distributed normal $N(0, t)$ with zero mean and variance $t$ we have that $\ln S_t$ follows the normal distribution with mean $\ln S_0 + (\mu - \frac{\sigma^2}{2}) t$ and variance $\sigma^2 t$. This implies by Equations (4) and (5) that $S_t$ follows the lognormal distribution with mean $S_0 e^{\mu t}$ and variance $S_0^2 e^{2\mu t} \left( e^{\sigma^2 t} - 1 \right)$. We can also integrate Equation (11) from $t$ to $T$ so that, analogous to Equation (12)
\[S_T = S_t \exp \left( (\mu - \frac{1}{2}\sigma^2) \tau + \sigma (W_T - W_t) \right) \quad (13)\]

and $S_T$ follows the lognormal distribution with mean $S_t e^{\mu \tau}$ and variance given by $S_t^2 e^{2\mu \tau} \left( e^{\sigma^2 \tau} - 1 \right)$. 

3
3.2 Bond Price

Apply Itô’s Lemma to the function \( \ln B_t \). Then \( \ln B_t \) follows the SDE

\[
d\ln B_t = r_t dt.
\]

Integrating from 0 to \( t \) we have

\[
d\ln B_t - d\ln B_0 = \int_0^t r_u du.
\]

so the solution to the SDE is \( B_t = \exp \left( \int_0^t r_u du \right) \) since \( B_0 = 1 \). When interest rates are constant then \( r_t = r \) and \( B_t = e^{rt} \). Integrating from \( t \) to \( T \) produces the solution \( B_{t,T} = \exp \left( \int_t^T r_u du \right) \) or \( B_{t,T} = e^{rT} \) when interest rates are constant.

3.3 Discounted Stock Price is a Martingale

We want to find a measure \( Q \) such that under \( Q \) the discounted stock price that uses \( B_t \) is a martingale. Write

\[
dS_t = r_t S_t dt + \sigma S_t dW^Q_t
\]

where \( W^Q_t = W_t + \frac{r_t - r}{\sigma^2} t \). We have that under \( Q \), at time \( t = 0 \), the stock price \( S_t \) follows the lognormal distribution with mean \( S_0 e^{r_t t} \) and variance \( S_0^2 e^{2r_t t} \left( e^{\sigma^2 t} - 1 \right) \), but that \( S_t \) is not a martingale. Using \( B_t \) as the numeraire, the discounted stock price is \( \tilde{S}_t = \frac{S_t}{B_t} \) and \( \tilde{S}_t \) will be a martingale. Apply Itô’s Lemma to \( \tilde{S}_t \), which follows the SDE

\[
d\tilde{S}_t = \frac{\partial \tilde{S}}{\partial B} dB_t + \frac{\partial \tilde{S}}{\partial S} dS_t
\]

since all terms involving the second-order derivatives are zero. Expand Equation (15) to obtain

\[
d\tilde{S}_t = \frac{-S_t}{B_t^2} dB_t + \frac{1}{B_t} dS_t
\]

\[
= \frac{-S_t}{B_t^2} (r_t B_t dt) + \frac{1}{B_t} \left( r_t S_t dt + \sigma S_t dW^Q_t \right)
\]

\[
= \sigma \tilde{S}_t dW^Q_t.
\]

The solution to the SDE (16) is

\[
\tilde{S}_t = \tilde{S}_0 \exp \left( -\frac{1}{2} \sigma^2 t + \sigma W^Q_t \right).
\]

This implies that \( \ln \tilde{S}_t \) follows the normal distribution with mean \( \ln \tilde{S}_0 - \frac{1}{2} \sigma^2 t \) and variance \( \sigma^2 t \). To show that \( \tilde{S}_t \) is a martingale under \( Q \), consider the expectation
under $\mathbb{Q}$ for $s < t$

$$E^Q \left[ \tilde{S}_t | F_s \right] = \tilde{S}_0 \exp \left( -\frac{1}{2} \sigma^2 t \right) E^Q \left[ \exp \left( \sigma W_t^Q \right) \right] \bigg| F_s$$

$$= \tilde{S}_0 \exp \left( -\frac{1}{2} \sigma^2 t + \sigma W_s^Q \right) E^Q \left[ \exp \left( \sigma (W_t^Q - W_s^Q) \right) \right] \bigg| F_s$$

At time $s$ we have that $W_t^Q - W_s^Q$ is distributed as $N(0, t - s)$ which is identical in distribution to $W_{t-s}$ at time zero. Hence we can write

$$E^Q \left[ \tilde{S}_t | F_s \right] = \tilde{S}_0 \exp \left( -\frac{1}{2} \sigma^2 t + \sigma W_s^Q \right) E^Q \left[ \exp \left( \sigma W_{t-s}^Q \right) \right] \bigg| F_0.$$ 

Now, the moment generating function (mgf) of a random variable $X$ with normal distribution $N(\mu, \sigma^2)$ is $E^Q e^{\phi X} = \exp(\mu \phi + \frac{1}{2} \phi^2 \sigma^2)$. Under $\mathbb{Q}$ we have that $W_t^Q$ is $\mathbb{Q}$-Brownian motion and distributed as $N(0, t - s)$. Hence the mgf of $W_{t-s}^Q$ is $E^Q \left[ \exp \left( \sigma W_{t-s}^Q \right) \right] = \exp \left( \frac{1}{2} \sigma^2 (t - s) \right)$ where $\sigma$ takes the place of $\phi$, and we can write

$$E^Q \left[ \tilde{S}_t | F_s \right] = \tilde{S}_0 \exp \left( -\frac{1}{2} \sigma^2 t + \sigma W_s^Q \right) \exp \left( \frac{1}{2} \sigma^2 (t - s) \right)$$

$$= \tilde{S}_0 \exp \left( -\frac{1}{2} \sigma^2 s + \sigma W_s^Q \right)$$

$$= \tilde{S}_s.$$ 

We thus have that $E^Q \left[ \tilde{S}_t | F_s \right] = \tilde{S}_s$, which shows that $\tilde{S}_t$ is a $\mathbb{Q}$-martingale.

Pricing a European call option under Black-Scholes makes use of the fact that under $\mathbb{Q}$, at time $t$ the terminal stock price at expiry, $S_T$, follows the normal distribution with mean $S_t e^{rt}$ and variance $S_t^2 e^{2rt} \left( e^{\sigma^2 r} - 1 \right)$ when the interest rate $r_t$ is a constant value, $r$. Finally, note that under the original measure the process for $\tilde{S}_t$ is

$$d\tilde{S}_t = (\mu - r) \tilde{S}_t dt + \sigma \tilde{S}_t dW_t$$

which is obviously not a martingale.

3.4 Summary

We start with the processes for the stock price and bond price

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$dB_t = r_t B_t dt.$$ 

We apply Itô’s Lemma to get the processes for $\ln S_t$ and $\ln B_t$

$$d\ln S_t = \left( \mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t$$

$$d\ln B_t = r_t dt,$$

which allows us to solve for $S_t$ and $B_t$

$$S_t = S_0 e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma W_t}$$

$$B_t = e^{r^* t + \sigma W_t}.$$
We apply a change of measure to obtain the stock price under the risk neutral measure $Q$

$$dS_t = rS_t + \sigma S_t dW^Q_t \quad \Rightarrow \quad S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W^Q_t}$$

Since $S_t$ is not a martingale under $Q$, we discount $S_t$ by $B_t$ to obtain $\tilde{S}_t = \frac{S_t}{B_t}$ and

$$d\tilde{S}_t = \sigma \tilde{S}_t dW^Q_t \quad \Rightarrow \quad \tilde{S}_t = \tilde{S}_0 e^{-\frac{1}{2}\sigma^2 t + \sigma W^Q_t}$$

so that $\tilde{S}_t$ is a martingale under $Q$. The distributions of the processes described in this section are summarized in the following table:

| Stochastic Process | Lognormal distribution for $S_T | \mathcal{F}_t$ | Process a martingale |
|--------------------|-----------------|------------------|
| $dS = \mu S dt + \sigma S dW$ | $S_t e^{\mu t}$ | $S_t^2 e^{2\mu t} e^{\sigma^2 t - 1}$ | No |
| $d\tilde{S} = \sigma \tilde{S} dW^Q$ with $\tilde{S} = \frac{S}{B}$ | $\tilde{S}_t$ | $\tilde{S}_t^2 e^{\sigma^2 t - 1}$ | Yes |
| $d\tilde{S} = (\mu - \sigma^2) S dt + \sigma S dW$ | $S_t e^{(\mu - \frac{1}{2}\sigma^2) t}$ | $S_t^2 e^{2(\mu - \frac{1}{2}\sigma^2) t} e^{\sigma^2 t - 1}$ | No |

This also implies that the logarithm of the stock price is normally distributed.

4 The Black-Scholes Call Price

In the following sections we show four ways in which the Black-Scholes call price can be obtained. Under a constant interest rate $r$ the time-$t$ price of a European call option on a non-dividend paying stock when its spot price is $S_t$ and with strike $K$ and time to maturity $\tau = T - t$ is

$$C(S_t, K, T) = e^{-rt} E^Q \left[ (S_T - K)^+ | \mathcal{F}_t \right]$$

which can be evaluated to produce Equation (1), reproduced here for convenience

$$C(S_t, K, T) = S_t \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

where

$$d_1 = \frac{\log \frac{S_t}{K} + \left( r + \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}$$

and

$$d_2 = d_1 - \sigma \sqrt{\tau} = \frac{\log \frac{S_t}{K} + \left( r - \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}.$$
The first derivation is by straightforward integration of Equation (17); the second is by applying the Feynman-Kac theorem; the third is by transforming the Black-Scholes PDE into the heat equation and solving the heat equation; the fourth is by using the Capital Asset Pricing Model (CAPM).

5 Black-Scholes by Straightforward Integration

The European call price $C(S_t, K, T)$ is the discounted time-$t$ expected value of $(S_T - K)^+$ under the EMM $Q$ and when interest rates are constant. Hence from Equation (17) we have

$$C(S_t, K, T) = e^{-rT} E^Q \left[ (S_T - K)^+ \mid F_t \right]$$

(18)

$$= e^{-rT} \int_K^{\infty} (S_T - K) dF(S_T)$$

$$= e^{-rT} \int_K^{\infty} S_T dF(S_T) - e^{-rT} K \int_K^{\infty} dF(S_T).$$

To evaluate the two integrals, we make use of the result derived in Section (3.3) that under $Q$ and at time $t$ the terminal stock price $S_T$ follows the lognormal distribution with mean $\ln S_t + \left( r - \frac{\sigma^2}{2} \right) \tau$ and variance $\sigma^2 \tau$, where $\tau = T - t$ is the time to maturity. The first integral in the last line of Equation (18) uses the conditional expectation of $S_T$ given that $S_T > K$

$$\int_K^{\infty} S_T dF(S_T) = E^Q [S_T \mid S_T > K]$$

$$= L_{S_t}(K).$$

This conditional expectation is, from Equation (10)

$$L_{S_t}(K) = \exp \left( \ln S_t + \left( r - \frac{\sigma^2}{2} \right) \tau + \frac{\sigma^2 \tau}{2} \right)$$

$$\times \Phi \left( \frac{-\ln K + \ln S_t + \left( r - \frac{\sigma^2}{2} \right) \tau + \sigma^2 \tau}{\sigma \sqrt{\tau}} \right)$$

$$= S_t e^{r\tau} \Phi(d_1),$$

so the first integral in the last line of Equation (18) is

$$S_t \Phi(d_1).$$

(19)
Using Equation (7), the second integral in the last line of (18) can be written

\[ e^{-rT} K \int_{K}^{\infty} dF(S_T) = e^{-rT} K \left[ 1 - F(K) \right] \]

\[ = e^{-rT} K \left[ 1 - \Phi \left( \frac{\ln K - \ln S_t - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right) \right] \]

\[ = e^{-rT} K \left[ 1 - \Phi (-d_2) \right] \]

\[ = e^{-rT} K \Phi (d_2). \]

Combining the terms in Equations (19) and (20) leads to the expression (1) for the European call price.

### 5.1 Change of Numeraire

The principle behind pricing by arbitrage is that if the market is complete we can find a portfolio that replicates the derivative at all times, and we can find an equivalent martingale measure (EMM) \( \mathbb{N} \) such that the discounted stock price is a martingale. Moreover, the EMM \( \mathbb{N} \) determines the unique numeraire \( N_t \) that discounts the stock price. The time-\( t \) value \( V(S_t, t) \) of the derivative with payoff \( V(S_T, T) \) at time \( T \) discounted by the numeraire \( N_t \) is

\[ V(S_t, t) = N_t E^{\mathbb{N}} \left[ \frac{V(S_T, T)}{N_T} \bigg| \mathcal{F}_t \right]. \]  

(21)

In the derivation of the previous section, the bond \( B_t = e^{rt} \) serves as the numeraire, and since \( r \) is deterministic we can take \( N_T = e^{rT} \) out of the expectation and with \( V(S_T, T) = (S_T - K)^+ \) we can write

\[ V(S_t, t) = e^{-r(T-t)} E^{\mathbb{N}} \left[ (S_T - K)^+ \bigg| \mathcal{F}_t \right] \]

which is Equation (17) for the call price.

### 5.1.1 Black Scholes Under a Different Numeraire

In this section we show that we can use the stock price \( S_t \) as the numeraire and recover the Black-Scholes call price. We start with the stock price process in Equation (14) under the measure \( \mathbb{Q} \) and with a constant interest rate

\[ dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}. \]

(22)

The relative bond price process is defined as \( \hat{B} = \frac{B}{S} \) and by It\={o}'s Lemma follows the process

\[ d\hat{B}_t = \sigma^2 \hat{B}_t dt - \sigma \hat{B}_t dW_t^{\mathbb{Q}}. \]

The measure \( \mathbb{Q} \) turns \( \hat{S} = \frac{S}{B} \) into a martingale, but not \( \hat{B} \). The measure \( \mathbb{P} \) that turns \( \hat{B} \) into a martingale is

\[ W_t^{\mathbb{P}} = W_t^{\mathbb{Q}} - \sigma t \]

(23)
so that 
\[ d\tilde{B}_t = -\sigma \tilde{B}_t dW^P_t \]

is a martingale under $\mathbb{P}$. The value of the European call is determined by using $N_t = S_t$ as the numeraire along with the payoff function $V(S_T, T) = (S_T - K)^+$ in the valuation Equation (21)

\[ V(S_t, t) = S_t E^P \left[ \frac{(S_T - K)^+}{S_T} \right] F_t \]

(24)

where $Z_t = \frac{1}{S_t}$. To evaluate $V(S_t, t)$ we need the distribution for $Z_T$. The process for $Z = \frac{1}{S}$ is obtained using Itô's Lemma on $S_t$ in Equation (22) and the change of measure in Equation (23)

\[ dZ_t = (r + \sigma^2) Z_t dt - \sigma Z_t dW^Q_t \]

\[ = -rZ_t dt - \sigma Z_t dW^P_t. \]

To find the solution for $Z_t$ we define $Y_t = \ln Z_t$ and apply Itô’s Lemma again, to produce

\[ dY_t = -\left(r + \frac{\sigma^2}{2}\right) dt - \sigma dW^P_t. \]

(25)

We integrate Equation (25) to produce the solution

\[ Y_T - Y_t = -\left(r + \frac{\sigma^2}{2}\right) (T - t) - \sigma (W^P_T - W^P_t). \]

so that $Z_T$ has the solution

\[ Z_T = e^{\ln Z_t - \left(r + \frac{\sigma^2}{2}\right)(T - t) - \sigma (W^P_T - W^P_t)}. \]

(26)

Now, since $W^P_T - W^P_t$ is identical in distribution to $W^P_T$, where $\tau = T - t$ is the time to maturity, and since $W^P_T$ follows the normal distribution with zero mean and variance $\sigma^2 \tau$, the exponent in Equation (26)

\[ \ln Z_t - \left(r + \frac{\sigma^2}{2}\right) (T - t) - \sigma (W^P_T - W^P_t), \]

follows the normal distribution with mean

\[ u = \ln Z_t - \left(r + \frac{\sigma^2}{2}\right) \tau = -\ln S_t - \left(r + \frac{\sigma^2}{2}\right) \tau \]

and variance $v = \sigma^2 \tau$. This implies that $Z_T$ follows the lognormal distribution with mean $e^{u + v/2}$ and variance $(e^v - 1) e^{2u + v}$. Note that $(1 - KZ_T)^+$ in the expectation of Equation (24) is non-zero when $Z_T < \frac{1}{K}$. Hence we can write this expectation as the two integrals

\[ E^P \left[ (1 - KZ_T) | F_t \right] = \int_{-\infty}^{\frac{1}{K}} dF_{Z_T} - K \int_{-\infty}^{\frac{1}{K}} Z_T dF_{Z_T} \]

(27)

\[ = I_1 - I_2 \]
where $F_{Z_T}$ is the cdf of $Z_T$ defined in Equation (7). The first integral in Equation (27) is

$$I_1 = F_{Z_T} \left( \frac{1}{K} \right) = \Phi \left( \frac{\ln \frac{1}{K} - u}{v} \right)$$

(28)

$$= \Phi \left( \frac{- \ln K + \ln S_t + \left( \frac{r + \sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}} \right)$$

$$= \Phi \left( d_1 \right).$$

Using the definition of $L_{Z_T}(x)$ in Equation (10), the second integral in Equation (27) is

$$I_2 = K \left[ \int_{-\infty}^{\infty} Z_T dF_{Z_T} - \int_{\frac{1}{K}}^{\infty} Z_T dF_{Z_T} \right]$$

(29)

$$= K \left[ E^p [Z_T] - L_{Z_T} \left( \frac{1}{K} \right) \right]$$

$$= K \left[ e^{u + v/2} - e^{u + v/2} \Phi \left( \frac{- \ln \frac{1}{K} + u + v}{\sqrt{v}} \right) \right]$$

$$= K e^{u + v/2} \left[ 1 - \Phi \left( \frac{- \ln S_t + \left( \frac{r - \sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}} \right) \right]$$

$$= \frac{K}{S_t} e^{-r \tau} \Phi \left( d_2 \right)$$

since $1 - \Phi (-d_2) = \Phi (d_2)$. Substitute the expressions for $I_1$ and $I_2$ from Equations (28) and (29) into the valuation Equation (24)

$$V(S_t, t) = S_t E^p \left[ (1 - KZ_T) | \mathcal{F}_t \right]$$

$$= S_t \left[ I_1 - I_2 \right]$$

$$= S_t \Phi \left( d_1 \right) - Ke^{-r \tau} \Phi \left( d_2 \right)$$

which is the Black-Scholes call price in Equation (1).

6 Black-Scholes From the Feynman-Kac Theorem

6.1 The Feynman-Kac Theorem

Suppose that $x_t$ follows the process

$$dx_t = \mu(x_t, t) dt + \sigma (x_t, t) dW_t^Q$$

(30)
and suppose the differentiable function \( V = V(x_t, t) \) follows the partial differential equation given by

\[
\frac{\partial V}{\partial t} + \mu(x_t, t) \frac{\partial V}{\partial x} + \frac{1}{2} \sigma(x_t, t)^2 \frac{\partial^2 V}{\partial x^2} - r(t, x)V(x_t, t) = 0
\]

(31)

with boundary condition \( V(x_T, T) \). The Feynman-Kac theorem stipulates that \( V(x_t, t) \) has solution

\[
V(x_t, t) = E^Q \left[ e^{-\int_t^T r(X_u, u) du} V(X_T, T) \bigg| \mathcal{F}_t \right].
\]

(32)

In Equation (32) the time-\( t \) expectation is with respect to the same measure \( Q \) under which the stochastic portion of Equation (30) is Brownian motion. See the Note on www.FRouah.com for illustrations of the Feynman-Kac theorem.

### 6.2 The Theorem Applied to Black-Scholes

To apply the Feynman-Kac theorem to the Black-Scholes call price, note that the value \( V_t = V(S_t, t) \) of a European call option written at time \( t \) with strike price \( K \) when interest rates are a constant \( r \) follows the Black-Scholes PDE

\[
\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV_t = 0
\]

(33)

with boundary condition \( V(S_T, T) = (S_T - K)^+ \). The Note on www.FRouah.com explains how the PDE (33) is derived. This PDE is the PDE in Equation (31) with \( x_t = S_t \), \( \mu(x_t, t) = rS_t \), and \( \sigma(x_t, t) = \sigma S_t \). Hence the Feynman-Kac theorem applies and the value of the European call is

\[
V(S_t, t) = E^Q \left[ e^{-\int_t^T r(X_u, u) du} V(S_T, T) \bigg| \mathcal{F}_t \right]
\]

(34)

which is exactly Equation (18). Hence, we can evaluate the expectation in (34) by straightforward integration exactly in the same way as in Section 5 and obtain the call price in Equation (1).

### 7 Black-Scholes From the Heat Equation

In this section we follow the derivation explained in Wilmott et al. [4]. We first present a definition of the Dirac delta function, and of the heat equation. We then transform the Black-Scholes PDE into the heat equation, apply the solution through integration, and convert back to the original (untransformed) parameters. This will produce the Black-Scholes call price.
7.1 Dirac Delta Function

The Dirac delta function $\delta(x)$ is defined as

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(\xi) d\xi = 1.$$

For an integrable function $f(x)$ we have that

$$f(0) = \int_{-\infty}^{\infty} f(\xi) \delta(\xi) d\xi$$

and

$$f(x) = \int_{-\infty}^{\infty} f(x - \xi) \delta(\xi) d\xi. \quad (35)$$

7.2 The Heat Equation

The heat equation is the PDE for $u = u(x, \tau)$ over the domain $\{x \in R, \tau > 0\}$ given by

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

The heat equation has the fundamental solution

$$u(x, \tau) = \frac{1}{\sqrt{4\pi \tau}} \exp \left( -\frac{x^2}{4\tau} \right) \quad (36)$$

which is the normal pdf with mean 0 and variance $2\tau$. The initial value of the heat equation is $u(x, 0) = u_0(x)$ which can be written in terms of the Dirac delta function $\delta$ as the limit

$$u_0(x) = \lim_{\tau \to 0} u(x, \tau) = \delta(x).$$

Using the property (35) of the Dirac delta function, we can write the initial value as

$$u_0(x) = \int_{-\infty}^{\infty} \delta(x - \xi) u_0(\xi) d\xi \quad (37)$$

We can also apply the property (35) to the fundamental solution in Equation (36) and express the solution as

$$u(x, \tau) = \int_{-\infty}^{\infty} u(x - \xi) \delta(\xi) d\xi = \int_{-\infty}^{\infty} u(x - \xi) u_0(\xi) d\xi = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \tau}} e^{-\frac{(x-\xi)^2}{4\tau}} u_0(\xi) d\xi,$$

with initial value

$$u(x, 0) = \int_{-\infty}^{\infty} \delta(x - \xi) u_0(\xi) d\xi = u_0(x),$$

as before.
7.3 The Black-Scholes PDE as the Heat Equation

Through a series of transformations we convert the Black-Scholes PDE in Equation (33) into the heat equation. The first set of transformations convert the spot price to log-moneyness and the time to one-half the total variance. This will get rid of the $S$ and $S^2$ terms in the Black-Scholes PDE. The first transformations are

$$x = \ln \frac{S}{K} \text{ so that } S = Ke^x \quad (39)$$

$$\tau = \frac{\sigma^2}{2} (T - t) \text{ so that } t = T - \frac{2\tau}{\sigma^2}$$

$$U(x, \tau) = \frac{1}{K} V(S,t) = \frac{1}{K} V(Ke^x, T - 2\tau/\sigma^2).$$

Apply the chain rule to the partial derivatives in the Black-Scholes PDE. We have

$$\frac{\partial U}{\partial t} = K \frac{\partial U}{\partial \tau} + \frac{1}{2} \sigma^2 K^2 e^{2x} \left( \frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x} \right) - rU = 0$$

Substitute for the partials in the Black-Scholes PDE (33) to obtain

$$\frac{-K \sigma^2 \partial U}{2} + r Ke^x e^{-x} \frac{\partial U}{\partial x} + \frac{1}{2} \sigma^2 K^2 e^{2x} e^{-2x} \left( \frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x} \right) - rU = 0$$

which simplifies to

$$\frac{-\partial U}{\partial \tau} + (k - 1) \frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2} - kU = 0 \quad (40)$$

where $k = \frac{2\tau}{\sigma^2}$. The coefficients of this PDE does not involve $x$ or $\tau$. The boundary condition for $V$ is $V(S_T, T) = (S_T - K)^+$. From Equation (39), when $t = T$ and $S_t = S_T$ we have that $x = \ln \frac{S_T}{K}$ which we write as $x_T$, and that $\tau = 0$. Hence the boundary condition for $U$ is

$$U_0(x_T) = U(x_T, 0) = \frac{1}{K} V(S_T - K)^+ = \frac{1}{K} (Ke^{x_T} - K)^+ = (e^{x_T} - 1)^+. $$
We make the additional transformation

\[ W(x, \tau) = e^{\alpha x + \beta^2 \tau} U(x, \tau) \]  

(41)

where \( \alpha = \frac{1}{2}(k-1) \) and \( \beta = \frac{1}{2}(k+1) \). This will convert Equation (40) into the heat equation. The partial derivatives of \( U \) in terms of \( W \) are

\[
\frac{\partial U}{\partial \tau} = e^{-\alpha x - \beta^2 \tau} \left( \frac{\partial W}{\partial \tau} - W(x, \tau) \beta^2 \right)
\]

\[
\frac{\partial U}{\partial x} = e^{-\alpha x - \beta^2 \tau} \left( \frac{\partial W}{\partial x} - \alpha W(x, \tau) \right)
\]

\[
\frac{\partial^2 U}{\partial x^2} = e^{-\alpha x - \beta^2 \tau} \left( \alpha^2 W(x, \tau) - 2\alpha \frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial x^2} \right).
\]

Substitute these derivatives into Equation (40) to obtain

\[
\beta^2 W(x, \tau) - \frac{\partial W}{\partial \tau} + (k - 1) \left[ -\alpha W(x, \tau) + \frac{\partial W}{\partial x} \right]
\]

\[+ \alpha W(x, \tau) - 2\alpha \frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial x^2} - kW(x, \tau) = 0\]

which simplifies to the heat equation

\[
\frac{\partial W}{\partial \tau} = \frac{\partial W^2}{\partial x^2}.
\]  

(42)

From Equation (41) the boundary condition for \( W(x, \tau) \) is

\[
W_0(x_T) = W(x_T, 0) = e^{\alpha x_T} U(x_T, 0)
\]

\[= \left( e^{(\alpha+1)x_T} - e^{\alpha x_T} \right)^+ = \left( e^{\beta x_T} - e^{\alpha x_T} \right)^+.
\]

since \( \beta = \alpha + 1 \). The transformation from \( V \) to \( W \) is therefore

\[
V(S, t) = \frac{1}{K} e^{-\alpha x - \beta^2 \tau} W(x, \tau).
\]  

(44)

### 7.4 Obtaining the Black-Scholes Call Price

Since \( W(x, \tau) \) follows the heat equation, it has the solution given by Equation (38), with boundary condition given by (43). Hence the solution is

\[
W(x, \tau) = \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4\tau} W_0(\xi) d\xi
\]

\[= \frac{1}{\sqrt{4\pi \tau}} \int_{-\infty}^{\infty} e^{-(\xi-x)^2/4\tau} \left( e^{\beta \xi} - e^{\alpha \xi} \right)^+ d\xi.
\]
Make the change of variable $z = \frac{\xi - x}{\sqrt{2\tau}}$ so that $\xi = \sqrt{2\tau}z + x$ and $d\xi = \sqrt{2\tau}dz$.

\[ W(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \exp\left(-\frac{1}{2}z^2\right) \times \exp\left(\beta \left[\sqrt{2\tau}z + x\right] - \alpha \left[\sqrt{2\tau}z + x\right]\right) dz \tag{45} \]

Note that the integral is non-zero only when the second exponent is greater than zero, that is, when $\beta \left[\sqrt{2\tau}z + x\right] > \alpha \left[\sqrt{2\tau}z + x\right]$ which is identical to $z > -\frac{x}{\sqrt{2\tau}}$.

We can now break up the integral into two pieces

\[ W(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \exp\left(-\frac{1}{2}z^2\right) \exp\left(\beta \left[\sqrt{2\tau}z + x\right]\right) dz \]
\[ - \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \exp\left(-\frac{1}{2}z^2\right) \exp\left(\alpha \left[\sqrt{2\tau}z + x\right]\right) dz \]
\[ = I_1 - I_2. \]

Complete the square in the first integral $I_1$. The exponent in the integrand is

\[-\frac{1}{2}z^2 + \beta \sqrt{2\tau}z + \beta x = -\frac{1}{2} \left(z - \beta \sqrt{2\tau}\right)^2 + \beta x + \beta^2 \tau.\]

The first integral becomes

\[ I_1 = e^{\beta x + \beta^2 \tau} \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(z - \beta \sqrt{2\tau})^2} dz. \]

Make the transformation $y = z - \beta \sqrt{2\tau}$ so that the integral becomes

\[ I_1 = e^{\beta x + \beta^2 \tau} \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau} - \beta \sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}y^2} dy \]
\[ = e^{\beta x + \beta^2 \tau} \left(1 - \Phi\left(-\frac{x}{\sqrt{2\tau}} - \beta \sqrt{2\tau}\right)\right) \]
\[ = e^{\beta x + \beta^2 \tau} \Phi\left(\frac{x}{\sqrt{2\tau}} + \beta \sqrt{2\tau}\right). \]

The second integral is identical, except that $\beta$ is replaced with $\alpha$. Hence

\[ I_2 = e^{\alpha x + \alpha^2 \tau} \Phi\left(\frac{x}{\sqrt{2\tau}} + \alpha \sqrt{2\tau}\right). \]

Recall that $x = \ln\frac{S}{R}$, $k = \frac{2\sigma}{\sigma^2}$, $\alpha = \frac{1}{2}(k - 1) = \frac{r - \sigma^2/2}{\sigma^2}$, $\beta = \frac{1}{2}(k + 1) = \frac{r + \sigma^2/2}{\sigma^2}$, and $\tau = \frac{1}{2}\sigma^2(T - t)$. Consequently, we have that

\[ \frac{x}{\sqrt{2\tau}} + \beta \sqrt{2\tau} = \frac{\ln\frac{S}{R} + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}} = d_1 \]

15
and that
\[
\frac{x}{\sqrt{2\tau}} + \alpha \sqrt{2\tau} = d_1 - \sigma \sqrt{T-t} = d_2.
\]

Hence the first integral is
\[I_1 = \exp (\beta x + \beta^2 \tau) \Phi (d_1).
\]
The second integral is identical except that \(\beta\) is replaced by \(\alpha\) and involves \(d_2\) instead of \(d_1\)
\[I_2 = \exp (\alpha x + \alpha^2 \tau) \Phi (d_2).
\]
The solution is therefore
\[W(x, \tau) = I_1 - I_2 = e^{\beta x + \beta^2 \tau} \Phi (d_1) - e^{\alpha x + \alpha^2 \tau} \Phi (d_2).
\]
The solution in Equation (46), expressed in terms of \(I_1\) and \(I_2\), is the solution for
\[W(x, \tau).
\] To obtain the solution for the call price \(V(S_t, t)\) we must use Equation (44) and transform the solution in (46) back to \(V\). From (44) and (46)
\[V(S_t) = Ke^{-\alpha x - \beta^2 \tau} W(x, \tau)
\]
\[= Ke^{-\alpha x - \beta^2 \tau} [I_1 - I_2].
\]
The first integral in Equation (47) is
\[Ke^{-\alpha x - \beta^2 \tau} e^{\beta x + \beta^2 \tau} \Phi (d_1) = Ke^{(\beta - \alpha) x} \Phi (d_1)
\]
\[= S \Phi (d_1)
\]
since \(\beta - \alpha = 1\). The second integral in Equation (47) is
\[Ke^{-\alpha x - \beta^2 \tau} e^{\alpha x + \alpha^2 \tau} \Phi (d_2) = Ke^{(\alpha^2 - \beta^2) \tau} \Phi (d_2)
\]
\[= Ke^{-r(T-t)} \Phi (d_2)
\]
since \(\alpha^2 - \beta^2 = -\frac{2r}{\sigma^2}\). Combining the terms in Equations (48) and (49) produces the Black-Scholes call price in Equation (1).

8 Black-Scholes From CAPM

8.1 The CAPM

The Capital Asset Pricing Model (CAPM) stipulates that the expected return of a security \(i\) in excess of the risk-free rate is
\[E[r_i] - r = \beta_i (E[r_M] - r)
\]
where \(r_i\) is the return on the asset, \(r\) is the risk-free rate, \(r_M\) is the return on the market, and
\[\beta_i = \frac{Cov[r_i, r_M]}{Var[r_M]}
\]
is the security’s beta.
8.2 The CAPM for the Assets

In the time increment $dt$ the expected stock price return, $E[r_S dt]$ is $E\left[\frac{dS_t}{S_t}\right]$, where $S_t$ follows the diffusion in Equation (2). The expected return is therefore

$$E\left[\frac{dS_t}{S_t}\right] = rd(t) + \beta_S (E[r_M] - r) dt.$$  \hfill (50)

Similarly, the expected return on the derivative, $E[r_V dt]$ is $E\left[\frac{dV_t}{V_t}\right]$, where $V_t$ follows the diffusion in (3), is

$$E\left[\frac{dV_t}{V_t}\right] = rd(t) + \beta_V (E[r_M] - r) dt.$$  \hfill (51)

8.3 The Black-Scholes PDE from the CAPM

Divide by $V_t$ on both sides of the second line of Equation (3) to obtain

$$\frac{dV_t}{V_t} = \frac{1}{V_t} \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} \frac{dS_t}{S_t} V_t,$$

which is

$$r_V dt = \frac{1}{V_t} \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} \frac{dS_t}{S_t} r_S dt.$$  \hfill (52)

Drop $dt$ from both sides and take the covariance of $r_V$ and $r_M$, noting that only the second term on the right-hand side of Equation (52) is stochastic

$$\text{Cov}[r_V, r_M] = \frac{\partial V}{\partial S} \frac{S_t}{V_t} \text{Cov}[r_S, r_M].$$

This implies the following relationship between the beta of the derivative, $\beta_V$, and the beta of the stock, $\beta_S$

$$\beta_V = \left( \frac{\partial V}{\partial S} \frac{S_t}{V_t} \right) \beta_S.$$  \hfill (53)

This is Equation (15) of Black and Scholes [1]. Multiply Equation (51) by $V_t$ to obtain

$$E[dV_t] = rV_t dt + V_t \beta_V (E[r_M] - r) dt$$ \hfill (54)

This is Equation (18) of Black and Scholes [1]. Take expectations of the second line of Equation (3), and substitute for $E[dS_t]$ from Equation (50)

$$E[dV_t] = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} [rS dt + S_t \beta_S (E[r_M - r]) dt] + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 dt.$$  \hfill (54)
Equate Equations (53) and (54), and drop $dt$ from both sides. The term involving $\beta_S$ cancels and we are left with

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV_t = 0.$$  \hspace{1cm} (55)

We recognize that Equation (55) is the Black-Scholes PDE in Equation (33). Hence, we can obtain the Black-Scholes call price by appealing to the Feynman-Kac theorem exactly as was done in Section (6.2) and solving the integral as in Section (5).

9 Incorporating Dividends

The Black-Scholes call price in Equation (1) is for a call written on a non-dividend-paying stock. There are two ways to incorporate dividends into the call price. The first is by assuming the stock pays a continuous dividend yield $q$. The second is by assuming the stock pays dividends in lump sums, "lumpy" dividends.

9.1 Continuous Dividends

We assume that the dividend yield $q$ is constant so that the holder of the stock receives an amount $qS_t dt$ of dividend in the time increment $dt$. After the dividend is paid out, the value of the stock drops by the dividend amount. In other words, without the dividend yield, the value of the stock increases by $rS_t dt$, but with the dividend yield the stock increases by $rS_t dt - qS_t dt = (r - q) S_t dt$. Hence, the expected return becomes $r - q$ instead of $r$, which implies that the risk-neutral process for $S_t$ follows Equation (14) but with drift $r - q$ instead of $r$

$$dS_t = (r - q) S_t dt + \sigma S_t dW^Q_t.$$  \hspace{1cm} (56)

Following the same derivation in Section (3), Equation (56) has solution

$$S_T = S_t \exp \left( \left( r - q - \frac{1}{2} \sigma^2 \right) \tau + \sigma W^Q_T \right)$$

where $\tau = T - t$. Hence, $S_T$ follows the lognormal distribution with mean $S_t e^{(r-q)\tau}$ and variance $S_t^2 e^{2(r-q)\tau} \left( e^{\sigma^2 \tau} - 1 \right)$. Proceeding exactly as in Equation (18), the call price is

$$C(S_t, K, T) = e^{-r\tau} L_{S_T}(K) - e^{-r\tau} \left[ 1 - F(K) \right].$$  \hspace{1cm} (57)
The conditional expectation \( L_{S_T}(K) \) from Equation (10) becomes

\[
L_{S_T}(K) = \exp \left( \ln S_t + \left( r - q - \frac{\sigma^2}{2} \right) \tau + \frac{\sigma^2 \tau}{2} \right) \times \Phi \left( \frac{-\ln K + \ln S_t + \left( r - q - \frac{\sigma^2}{2} \right) \tau + \sigma^2 \tau}{\sigma \sqrt{\tau}} \right)
\]

(58)

with \( d_1 \) redefined as

\[
d_1 = \frac{\ln S_t + \left( r - q + \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}.
\]

Using Equation (7), the second term in Equation (18) becomes

\[
e^{-r\tau} K \left[ 1 - F(K) \right] = e^{-r\tau} K \left[ 1 - \Phi \left( \frac{\ln K - \ln S_t - \left( r - q - \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}} \right) \right]
\]

(59)

with \( d_2 = d_1 - \sigma \sqrt{\tau} \) as before. Substituting Equations (58) and (59) into Equation (57) produces the Black-Scholes price of a European call written on a stock that pays continuous dividends

\[
C(S_t, K, T) = S_t e^{-qT} \Phi(d_1) - e^{-rT} K \Phi(d_2).
\]

Hence, the only modification is that the current value of the stock price is decreased by \( e^{-qT} \), and the return on the stock is decreased from \( r \) to \( r - q \). All other computations are identical.

### 9.2 Lumpy Dividends

To come. Same idea: the current value of the stock price is decreased by the dividends, except not continuously.

### References


