FX Basket Options

Master Thesis

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Abstract

Although, we refute the iid normality assumption for daily and weekly FX returns, we give different approaches to price and hedge mainly European arithmetic as well as geometric basket options under the assumption of iid normal distributed returns. The main goal is to conserve consistency with the initial Black-Scholes assumption of underlying assets follow geometric Brownian motions. On one hand we fit the statistical properties of an arithmetic basket to known distributions where we derive closed-form results for the option price. On the other hand we give a numerical procedure, based on finite differences, to price European as well as American basket options. As an extension, we introduce so-called Wishart processes from where we derive closed-form solutions for geometric European basket options with stochastic covariance.

Contents

1 Introduction **4**

2 Properties of FX Returns **6**

3 Different Approaches to Price Basket Options **8**
3.1 Geometric Basket . . . 8
3.2 Closed-Form Approximation for Valuing an Arithmetic Basket Option . . . 10
3.2.1 Numerical Example to closed-form Approximations . . . 17
3.3 A Numerical Approach with Finite Differences . . . 18
3.3.1 Meshfree Numerical Approximation Method . . . 21
3.3.2 Numerical Results . . . 24

4 Hedging of Basket Options **25**
4.1 Geometric Basket . . . 26
4.2 Arithmetic Basket with the Closed-Form Approximations . . . 29
4.2.1 Greeks of the Lognormal Approximation . . . 29
4.2.2 A Static Super-Replicating Strategy . . . 32
4.2.3 A Static Sub-Replicating Strategy . . . 34

5 Stochastic Correlation in an Arithmetic Basket Option **35**
5.1 Calibration to Data . . . 37
5.2 Pricing an Arithmetic Basket with Stochastic Correlation . . . 39
5.3 Wishart Multi-Dimensional Stochastic Volatility . . . 40
5.3.1 Pricing a Geometric Basket Option with Stochastic Covariance . . . 41
5.4 Calibration data to the Wishart Process . . . 47
5.4.1 Calibrating Data to a five dimensional Basket of Currency Rates . . . 50
5.4.2 Influence of Stochastic Covariance on the Price of an Arithmetic Basket Call Option . . . 51
5.4.3 Introducing Correlation between the Underlying Assets and the Wishart Process . . . 52

6 Conclusions **53**

References **55**

A Appendix **57**
A.1 Degrees of Freedom of a Student t Fit . . . 57
A.2 The HKKP Tail-Index Estimator . . . 58
A.3 Autocorrelation in Return Series for Different Currencies . . . 60

1 Introduction

The financial industry has currently a strong interest in so-called basket options, which are multidimensional derivatives with respect to a chosen strike. This paper is mainly devoted to the foreign exchange (FX) market. The reasons to hold a FX basket option are different. In case of an investor with different FX exposures, think of a multinational company, a basket option can be a much cheaper instrument to hedge the position compared to other hedging instruments. For example, if the investor would hedge each FX exposure by a single underlying option, the transaction costs would increase in the number of currencies he has to hedge. In contrast, the basket option allows to hedge all FX exposures with one derivative. Therefore, the transaction cost are comparatively small. Another reason to hold a basket option are correlation effects. As we will see later, a basket option can be super-replicated by a portfolio of single underlying options. This portfolio does not profit by the correlation between the single underlying assets. On the other hand, a basket options does this. Hence, the price of the derivative is reduced. In this paper, the basket can be structured either as a weighted product (geometric basket) or as a weighted sum (arithmetic basket) of underlying assets. Weighted products of lognormal random variables are lognormal. Therefore, one can price geometric basket options with the same methodology as for single underlying assets. But, it is a well-known fact, a sum of lognormal distributed random variables is not lognormal. Many people ignore this point and assume the Black-Scholes formula, see [Black and Scholes, 1973], to price basket options, which leads obviously to an inconsistency in the models. We give here different approaches to price and hedge mainly European basket options with respect to the Black-Scholes assumption of the underlying assets. On the other side, we give an approach to overcome the weakness of constant volatility in the Black-Scholes model by introducing Wishart processes. These processes allow multidimensional stochastic covariance structures, which is an extension of models with constant covariance. They pay attention to the fact, that correlations and covariances change over time.

Monte Carlo (MC) simulations are an easy but expensive way to price derivatives. Especially in the case of multidimensional options the number of simulations increases enormously and become slow. In [Milevsky and Posner, 1998a], [Milevsky and Posner, 1998b] and [Milevsky and Posner, 1999] we find a different approach, which leads to closed form approximations of arithmetic European basket option. They identify the statistical properties of an arithmetic basket with other distributions. In two cases they need only the first two moments to match the moments to a lognormal or an inverse gamma distribution. In the third case, they match the first four moments of the basket to a distribution from the Johnson family, introduced in [Johnson, 1949]. Another approach is to price the option numerically by finite elements or finite differences. The Black-Scholes partial differential equation (PDE) can be generalized straightforward to the multidimensional case. The properties of the infinitesimal generator remain the same. In [Fasshauer et al., 2004] we find a method, which uses finite differences to price American options. They add a penalty term to the multidimensional Black-Scholes PDE to extend the European to the American problem. The idea of the penalty term follows [Nielsen et al., 2002]. The hedging request for arithmetic baskets is given in [Su, 2005]. He presents an approach of hedging the basket with principal components and a super-replicating strategy. Wishart processes are introduced in [Bru, 1991]. Some properties of this paper are used by [Fonseca et al., 2005] to price a single underlying European option with stochastic variance. The calibration of the Wishart processes is covered in [Gourieroux et al., 2004].

In Section 2 we analyze the properties of historical FX returns. We implement three tests to verify the iid normality assumption of the returns. The main result is that the iid normality assumption does not hold for daily and weekly returns, but it is supposable for monthly returns. In Section 3 we give different approaches to price basket options with the lognormal assumption of the underlying assets. We use the fact that infinite sums of correlated lognormal distributed random variables are inverse gamma distributed. Therefore, we match the arithmetic basket to different distributions like the inverse gamma, the lognormal and a distribution from the Johnson family by identifying the moments. This leads to closed-form pricing formulas for European basket options. Additionally, we give a numerical method, based on finite differences on nonlinear parabolic PDE, to price European and American style basket options. The obtained algorithm is quite simple, but the complexity of the calculations increases in the dimension of the basket. Therefore, we obtain only results for a two dimensional example. In Section 4 we consider the problem of hedging basket options. Two ideas are used: Dynamic hedging via Greeks and static hedging. In the later case we give a sub- and super-replicating strategy. Section 5 is devoted to stochastic covariance in a multidimensional framework. Historical analysis of correlation data shows that there is a stochastic component which can be described by a mean-reverting process. In a multidimensional basket we face the constrained of positive semi-definite covariance matrices. We overcome exactly this problem by introducing Wishart processes. The obtained stochastic process for the covariance matrix fulfills the property of positive semi-definiteness at any time. Additionally, we adapt an approach given in [Fonseca et al., 2005] to price geometric basket options with stochastic covariance. In fact, Wishart processes are multidimensional Ornstein-Uhlenbeck processes. Finally, we devote an extra part to calibration.

2 Properties of FX Returns

In many application and many papers people use the assumption of iid normal distributed returns. We check in this section whether and when this assumption holds. We make three tests to indicate how good this assumption is.

In the first test we fit returns of FX market data for different currencies to a Student t distribution. It is well known, that a normal distribution is equivalent to a Student t distribution with infinite many degrees of freedom. Therefore, we fit market FX rates from January $4^{th}, 1999$ until April $27^{th}, 2006$ by maximum likelihood to a Student t distribution. In Appendix A.1 we find information about the Student t distribution and the results from the fit. The main result is that the returns are more likely to be normally distributed for longer tenors. We calculate ν, the degrees of freedom of the Student t distribution, for daily, weekly, and monthly returns. In most cases, the returns have quite large degrees of freedom for monthly data, which indicates that the normal assumptions could hold. For daily returns we have in most cases not more than five or six degrees of freedom and therefore not more than four or five existing moments. Note, that a normal distribution have infinite many existing moments and that the $\nu - 1$ is the largest existing moment of a Student t distribution. In a second step we analyze the information in the tails of the distribution. Hill proposed in [Hill, 1975] a tail index γ to estimate the fat-tailedness of empirical data. Unfortunately, this estimator leads for small samples to non-satisfying results. Huisman, Koedijk, Kool and Palm states in [Huisman et al., 2001] that the tail index γ is approximately linear in the sample size k. In Appendix A.2 we give the details, how one calculates the HKKP tail index. Furthermore, we analyze the same set of FX market data as above and find the same results. Again, we calculate the tail-index for different tenors (daily, weekly, monthly) and obtain that the tail-index γ decreases with increasing tenor. This indicates that the normality assumption becomes more suitable for a longer tenor.

Finally, we plot autocorrelation diagrams of the FX market data returns. We want to check, how good the iid assumption is, which we mention above. Interestingly, we see for all tenors (daily, weekly, monthly) less autocorrelation. But if the returns are really iid then also the absolute returns should show less autocorrelation. This does not hold in many cases. Again, we remark, that the autocorrelation effect for the absolute returns decreases with increasing tenor. Hence, we have a further indication that the normality assumption for monthly returns could be true. Additionally, we plot the autocorrelation diagrams for different basket. One remarks, that the autocorrelation effect decreases with local diversification. That means, that the basket with FX rates from all parts of the world has less absolute autocorrelation than a basket of FX rates from a certain part of the world. The details for the calculation are given in Appendix A.3.

All three test indicate the normality assumption could hold for monthly returns and that it is probably wrong for shorter periods. Since we want to generalize the Black-Scholes assumption to multidimensional baskets, we use normal distributed returns for all tenors and derive consistent results for higher dimensions.

3 Different Approaches to Price Basket Options

In this section we give different approaches to price and hedge basket options. Throughout the whole paper, if it is not indicated explicitly, we use the following dynamic for the underlying assets:

$$dS_i = S_i r_i dt + S_i \sigma_i dW_i \qquad i = 1, .., n, \tag{3.1}$$

where r_i and σ_i are real constants and W_i is a standard Brownian motion satisfying

$$< dW_i, dW_j >= \varrho_{ij} dt.$$

We derive three ways to price basket options. First, we consider a weighted product of the underlying assets versus a constant strike. This derivative is called geometric basket. This is the simplest way to structure a basket and we solve the pricing problem analogue to the single underlying problem of Black-Scholes. In a second approach we match the moments of an arithmetic basket, which is a weighted sum of underlying assets, to some specific distributions. This leads to closed-form solutions and overcomes the problem that the distribution of an arithmetic basket is unknown. Finally, we derive solutions for European and American arithmetic basket options by finite differences. This approach is extended to exotic knock-out options.

3.1 Geometric Basket

Finite products of lognormal random variables are again lognormal. This leads to structuring a basket as a geometric composition of underlying assets and using standard methods from the one dimensional Black-Scholes model. Let the basket be given as

$$B(t) = \prod_{i=1}^{n} S_i(t)^{a_i} \qquad a_i \in \mathbb{R}_{++}, \quad i = 1, .., n. \tag{3.2}$$

Considering an European call with strike K on the basket $B(t)$, we obtain the price $C(B(0), 0)$ as

$$C(B(0), 0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(B(T) - K)^{+}\right],$$

under the risk neutral measure $\mathbb{Q}$. We determine r at the end of this section. Let us write $C(B(0), 0)$ with the indicator function as

$$\begin{aligned} e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(B(T) - K)^{+}\right] &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(B(T) - K)\mathbf{1}_{B(T)>K}\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[B(T)\mathbf{1}_{B(T)>K}\right] - Ke^{-rT}\mathbb{Q}(B(T) > K). \end{aligned}$$

As already indicated, a finite sum of normal random variables is normal and therefore we write the product of the underlying assets in the following way:

$$\begin{aligned} \prod_{i=1}^{n} S_i(T)^{a_i} &= \prod_{i=1}^{n} S_i(0)^{a_i} e^{a_i(r_i - \frac{\sigma_i^2}{2})T + a_i\sigma_i W_i(T)} \\ &= B(0)e^{\sum_{i=1}^{n} a_i(r_i - \frac{\sigma_i^2}{2})T + \sum_{i=1}^{n} a_i\sigma_i W_i(T)} \\ &= B(0)e^{aT + \sigma W(T)}, \end{aligned}$$

where $a := \sum_{i=1}^{n} a_i(r_i - \frac{\sigma_i^2}{2})$ and $\sigma := \sqrt{\sum_{i,j=1}^{n} a_i a_j \varrho_{ij}\sigma_i\sigma_j}$. Here, $\sum_{i=1}^{n} a_i\sigma_i W_i(T)$ is normal distributed with mean zero and variance $\sigma^2 T$. Therefore, we reduce the original problem to a standard Black-Scholes problem. The price follows by

$$C(B(0), 0) = e^{-rT}\left(B(0)e^{aT + \frac{\sigma^2}{2}T}\Phi(d_2) - K\Phi(d_1)\right),$$

where $d_1 := \frac{\log\frac{B(0)}{K} + aT}{\sigma\sqrt{T}}$, $d_2 := d_1 + \sigma\sqrt{T}$ and $\Phi(\cdot)$ is the cumulative normal distribution function (CDF). There is still the question what is r? By using Itô's formula we obtain the $\mathbb{Q}$-dynamics of $B(t)$ by

$$\frac{dB(t)}{B(t)} = (a + \frac{\sigma^2}{2})dt + \sigma dW(t).$$

Hence, the $\mathbb{Q}$-drift r is given by $a + \frac{\sigma^2}{2}$ and

$$C(B(0), 0) = B(0)\Phi(d_2) - e^{-rT} K\Phi(d_1). \tag{3.3}$$

3.2 Closed-Form Approximation for Valuing an Arithmetic Basket Option

Motivated by the fact that sums of lognormal random variables are not lognormal, we derive moment based approximations to price arithmetic basket options. We demonstrate three approximations which lead to closed-form pricing formulas based on moment matching. First, we approximate the returns of the basket by a lognormal functional form. Secondly, we approximate the basket by an inverse-gamma functional form and finally we use a Johnson approximation for the unknown density function of the basket returns. In the second approach we are inspired by the result that an infinite sum of correlated lognormal random variables is inverse gamma distributed. The first two approaches use only the first two moments while the Johnson approximation uses further moments up to order 4. One can argue that through higher moments the Johnson approximation leads to better results than the other approximations. On the other side we know that distributions from the Johnson family put positive probability weight to negative values of the random variable, what is obviously not desirable for a basket. In a numerical example we see that the inverse gamma approximation leads to similar results than the Johnson approximation and the errors compared to a MC simulation are from the same order in the chosen constellation.

Remark 3.1 *For calculating the moments of a lognormal distributed random variable, we use that*

$$M_n := \mathbb{E}[X^n] = e^{n\mu + \frac{1}{2}n^2\sigma^2}, \qquad n = 1, 2, 3, ...$$

for $X \sim LN(\mu, \sigma)$.

Let us assume we have a basket given by

$$B(t) = \sum_{i=1}^{n} w_i S_i(t),$$

where $\mathbf{w} = (w_1, \ldots, w_n)'$ are the weights in the underlying assets. In a complete market the unique no-arbitrage price of an European basket call option is

$$\begin{aligned} V_{call} &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(B(T)-K)^+\right] \\ &= e^{-rT}\int_0^\infty (B(T)-K)^+ d\Psi[B(T)], \end{aligned}$$

where $\Psi[B(T)]$ denotes the state-price density (SPD). Note, that the SPD is not known in general and closed-form solutions are not available for the chosen dynamic in (3.1) of the underlying assets. An approach to approximate the integral is to assume that the SPD has a lognormal functional form. Therefore, we match the moments of the basket $B(T)$ to a lognormal distribution. We define the "pseudo-forward" of the basket as

$$F = \sum_{i=1}^{n} w_i F_i,$$

where

$$F_i = S_i(0)e^{r_i T}.$$

Hence, the first moment of the normalized basket $B^*(T) = B(T)/F$ is 1, or $M_1^* = 1$. The second moment can be found in [Milevsky and Posner, 1998a] or can be obtained by straightforward calculations. It is

$$M_2^* = \frac{1}{F^2}\sum_{i,j=1}^{n} w_i w_j F_i F_j e^{\varrho_{ij}\sigma_i\sigma_j T}.$$

We obtain the following proposition by assuming the normalized basket $B^*(T)$ is lognormal:

Proposition 3.2 *The no-arbitrage price of an European basket call option at time $t = 0$ is given by*

$$C_{call,LN} \cong e^{-rT}\left[F\Phi\left(\frac{\log(F/K)+v/2}{\sqrt{v}}\right) - K\Phi\left(\frac{\log(F/K)-v/2}{\sqrt{v}}\right)\right],$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function (CDF) and $v := \log M_2^*$.

Proof: By assuming that $B(T)/F \sim LN(0, v)$, we get

$$B(T) = Fe^{-\frac{v}{2}+\sqrt{v}\epsilon},$$

where $\epsilon \sim N(0,1)$. Therefore, we calculate $\mathbb{E}[(B(T)-K)^+]$ as in the Black-Scholes formula by applying Girsanov's Theorem and discount the result by e^{-rT}. See [Milevsky and Posner, 1998a] for further details. $\square$

A similar approach is to match moments to an inverse gamma instead of a lognormal distribution. This approach is justified by the result that an infinite sum of correlated lognormal random variables is inverse gamma distributed. Let us first give a remark about the definition and some properties of a gamma respectively inverse-gamma distribution. Then, the next proposition, in which we derive the price of a basket call option by assuming the basket B to be inverse-gamma distributed, follows analogue as the lognormal approximation.

Remark 3.3 *If a random variable X is gamma distributed, $X \sim Gamma(\alpha, \beta)$, the probability density function (PDF) of X is given by*

$$g(x, \alpha, \beta) = \frac{e^{-x/\beta}(x\beta)^{\alpha-1}}{\beta\Gamma(\alpha)}, \quad x \geq 0, \quad \alpha, \beta > 0.$$

If a random variable Y is inverse-gamma distributed, $Y \sim IG(\alpha, \beta)$, then its inverse $1/Y$ is gamma distributed. We use the notation $G_I(y, \alpha, \beta)$ and $G(x, \alpha, \beta)$ to denoted the CDF of the inverse gamma respectively gamma distribution evaluated at y and x. By definition we have

$$G_I(y, \alpha, \beta) = 1 - G(1/y, \alpha, \beta).$$

The probability density functions are related by

$$g_I(y, \alpha, \beta) = \frac{g(1/y, \alpha, \beta)}{y^2}, \quad y \geq 0, \quad \alpha, \beta > 0.$$

The moments of $Y \sim IG(\alpha, \beta)$ *are given by*

$$\mathbb{E}[Y^n] = \frac{1}{\beta^n(\alpha-1)(\alpha-2)...(\alpha-n)}, \quad n = 1, 2, 3, \dots. \tag{3.4}$$

Let us assume $B(T) \sim IG(\alpha, \beta)$ and let M_1, M_2 denote the first and second moment of $B(T)$. We obtain a second closed-form approximation for the arithmetic European basket call option in the following proposition.

Proposition 3.4 *If* $B(T) \sim IG(\alpha, \beta)$ *the no-arbitrage value of an arithmetic basket call option at time* $t = 0$ *is*

$$C_{call,IG} \cong e^{-rT}\left[M_1 G\left(\frac{1}{K}, \alpha - 1, \beta\right) - KG\left(\frac{1}{K}, \alpha, \beta\right)\right],$$

where $\alpha = (2M_2 - M_1^2)/(M_2 - M_1^2)$ *and* $\beta = (M_2 - M_1^2)/(M_2 M_1)$.

Proof: It is not difficult to show that

$$\alpha = \frac{2M_2 - 1}{M_2 - 1},$$
$$\beta = 1 - \frac{1}{M_2},$$

by assuming $B^* \sim IG(\alpha, \beta)$ and by (3.4). Then, we use Remark 3.3 to calculate

$$\begin{aligned}
\mathbb{E}[(B(T) - K)^+] &= F\int_0^\infty (B^* - K/F)^+ g_I(B^*, \alpha, \beta)dB^* \\
&= F\int_{K/F}^\infty x g_I(x, \alpha, \beta)dx - K\int_{K/F}^\infty g_I(x, \alpha, \beta)dx \\
&= F\int_{K/F}^\infty \frac{g(1/x, \alpha, \beta)}{x}dx - K\int_{K/F}^\infty \frac{g(1/x, \alpha, \beta)}{x^2}dx \\
&= F\int_0^{F/K} \frac{g(u, \alpha, \beta)}{u}du - K\int_0^{F/K} g(u, \alpha, \beta)du \\
&= F\int_0^{F/K} g(u, \alpha - 1, \beta)du - K\int_0^{F/K} g(u, \alpha, \beta)du \\
&= FG(F/K, \alpha - 1, \beta) - KG(F/K, \alpha, \beta).
\end{aligned}$$

Here, we calculate the second last equality from

$$\frac{g(x, \alpha, \beta)}{x} = g(x, \alpha - 1, \beta).$$

The proof follows by discounting the result by e^{-rT}. See [Milevsky and Posner, 1999] for further details. $\square$

So far we used only the first two moments to match the distribution of $B(T)$ to other distributions from where we can derive closed-form solutions for the price of an arithmetic basket option. We find an approach which uses higher moments up to order 4 in [Milevsky and Posner, 1998b]. They approximate the statistical properties of the basket $B(T)$ by a SPD from the Johnson family introduced in [Johnson, 1949]. The benefit and rationale for using the Johnson family is twofold. First, the family is rich enough to accommodate a wide variety of lognormal-like distributions. Second, the price of an arithmetic basket option can be solved analytically. The Johnson family, which we will present, has four degrees of freedom and can thus be identically matched to the first four moments of $B(T)$. It is a collection of statistical distributions, parameterized by four variables that can be represented by a transformation of the standard normal distributed random variable $Z \sim N(0,1)$ in the following way:

$$X = c + d\phi^{-1}\left(\frac{Z - a}{b}\right).$$

The function ϕ can be quite general, but is usually restricted to $\ln(\cdot)$ (Type I) or $\sinh^{-1}(\cdot)$ (Type II). In [Milevsky and Posner, 1998b] is stated that for the Type I formulation the skewness parameter forces the kurtosis parameter. Therefore, we use here Type II formulation and approximate the basket $B(T)$ by

$$B(T) \cong c + d \sinh\left(\frac{Z - a}{b}\right). \tag{3.5}$$

After we have derived the moments of $B(T)$ in (3.5), we have to solve a nonlinear system of equations after a, b, c, and d. The system of equations is given by:

$$
\begin{aligned}
M_1 &= c - de^{\frac{1}{2b^2}} \sinh \frac{a}{b}, \\
M_2 &= c^2 + \frac{d^2}{2}(e^{\frac{2}{b^2}} \cosh \frac{2a}{b} - 1) - 2cde^{\frac{1}{2b^2}} \sinh \frac{a}{b}, \\
M_3 &= c^3 - 3c^2 de^{\frac{1}{2b^2}} \sinh \frac{a}{b} + \frac{3}{2} cd^2 (e^{\frac{2}{b^2}} \cosh \frac{2a}{b} - 1) \\
&\quad + \frac{d^3}{4}(3e^{\frac{1}{2b^2}} \sinh \frac{a}{b} - e^{\frac{9}{2b^2}} \sinh \frac{3a}{b}), \\
M_4 &= c^4 - 4c^3 de^{\frac{1}{2b^2}} \sinh \frac{a}{b} + 3c^2 d^2 (e^{\frac{2}{b^2}} \cosh \frac{2a}{b} - 1) \\
&\quad + cd^3 (3e^{\frac{1}{2b^2}} \sinh \frac{a}{b} - e^{\frac{9}{2b^2}} \sinh \frac{3a}{b}) \\
&\quad + \frac{d^4}{8}(e^{\frac{8}{b^2}} \cosh \frac{4a}{b} - 4e^{\frac{2}{b^2}} \cosh \frac{2a}{b} + 3).
\end{aligned}
$$

This system can be solved by numerical solvers, but there is no symbolic solution for a, b, c, and d. Finally, we obtain a third closed-form approximation of the European basket call by the following proposition.

Proposition 3.5 *The no-arbitrage price of an European basket call at time $t = 0$ is approximated by*

$$
C_{call,Johnson} \cong e^{-rT} \left[M_1 - K + (K - c)\Phi(Q) + \frac{d}{2} e^{\frac{1}{2b^2}} \left(e^{\frac{a}{b}} \Phi\left(Q + \frac{1}{b}\right) - e^{-\frac{a}{b}} \Phi\left(Q - \frac{1}{b}\right) \right) \right],
$$

where $Q := a + b \sinh^{-1}\left(\frac{K-c}{d}\right)$ and $\Phi(.)$ is the standard normal CDF.

Proof: Let us denote by $h(x)$ the PDF and by $H(x)$ the CDF of $B(T)$. The price of a basket call option is given as the discounted value of

$$\begin{aligned}
\mathbb{E}[(B(T)-K)^+] &= \int_K^\infty (x-K)h(x)dx \\
&= \int_0^\infty xh(x)dx - K\int_0^\infty h(x)dx - \int_0^K (x-K)h(x)dx \\
&= M_1 - K - \int_0^K (x-K)h(x)dx \\
&= M_1 - K - \left[(x-K)\int_0^x h(z)dz\right]\Big|_0^K + \int_0^K \left[\int_0^x h(z)dz\right]dx \\
&= M_1 - K + \int_0^K H(x)dx.
\end{aligned}$$

Here, we integrate partially. Finally, we approximate $\int_0^K H(x)dx$ by $\int_{-\infty}^K H(x)dx$. We find detailed calculations in [Milevsky and Posner, 1998b] and obtain

$$\int_{-\infty}^K H(x)dx = (K-c)\Phi(Q) + \frac{d}{2}e^{\frac{1}{2b^2}}\left(e^{\frac{a}{b}}\Phi\left(Q+\frac{1}{b}\right) - e^{-\frac{a}{b}}\Phi\left(Q-\frac{1}{b}\right)\right),$$

from where the proof follows. □

Remark 3.6 *The function* $\sinh(\cdot)$ *has a range on the real numbers, which leads to a criticism since the basket* $B(T)$ *is defined only on the positive real numbers. The lognormal and the inverse-gamma approximation do not face this weakness. But [Milevsky and Posner, 1999] state that* $\int_{-\infty}^0 H(x)dx \approx 0$ *after calibrating* a*,* b*,* c*, and* d*. Therefore, we put only small probability to the negative real numbers. We see in the example of Section 3.2.1 that exactly this holds and that the mentioned weakness of the Johnson approximation is compensated by the higher moments which are used there. For completeness, we give the first four moments*

Moneyness	MC	MC Std	LN	Rel err	IG	Rel err	Johnson	Rel err
0.85	19.0216	0.0466	19.1162	5.0e-03	19.0446	1.2e-03	19.0468	1.3e-03
0.95	11.0902	0.0511	11.1985	9.8e-03	11.1030	1.2e-03	11.0497	3.6e-03
1.00	7.9159	0.0373	8.0491	1.68e-02	7.9776	7.8e-03	7.9082	9.98e-04
1.05	5.4579	0.0307	5.5367	1.44e-02	5.5063	8.9e-03	5.4426	2.8e-03
1.15	2.3381	0.0191	2.3006	1.60e-02	2.3480	4.2e-03	2.33608	8.72e-04

Table 1: Results of the approximations derived in Propositions 3.2 - 3.5 for a three dimensional basket. The parameters are chosen as $t = 0$, $T = 1$, $\mathbf{S}(0) = (25, 30, 45)'$, $\boldsymbol{\sigma} = (0.20, 0.40, 0.30)'$, $\mathbf{w} = (1, 1, 1)'$ and $\mathbf{r} = (0.02, 0.04, 0.06)'$. The time interval from 0 to 1 is split into 250 subintervals. The MC simulation is the mean and the standard deviation of 10 simulations with 100'000 different paths. The moneyness is defined by $K/B(0)$ where K is the strike. The correlation structure is chosen as $\varrho_{12} = 0.5$ and $\varrho_{13} = \varrho_{23} = -0.5$.

of $B(T)$, which are used in the nonlinear system above.

$$
\begin{aligned}
M_1 &= \sum_{i=1}^{n} w_i F_i, \\
M_2 &= \sum_{i,j=1}^{n} w_i w_j F_i F_j \exp\left(\sigma_i \sigma_j \varrho_{ij} T\right), \\
M_3 &= \sum_{i,j,k=1}^{n} w_i w_j w_k F_i F_j F_k \exp\left(\sigma_i \sigma_j \varrho_{ij} T + \sigma_i \sigma_k \varrho_{ik} T + \sigma_j \sigma_k \varrho_{jk} T\right), \\
M_4 &= \sum_{i,j,k,l=1}^{n} w_i w_j w_k w_l F_i F_j F_k F_l \cdot \\
&\quad \cdot \exp\left(\sigma_i \sigma_j \varrho_{ij} T + \sigma_i \sigma_k \varrho_{ik} T + \sigma_i \sigma_l \varrho_{il} T + \sigma_j \sigma_k \varrho_{jk} T + \sigma_j \sigma_l \varrho_{jl} T + \sigma_k \sigma_l \varrho_{kl} T\right).
\end{aligned}
$$

3.2.1 Numerical Example to closed-form Approximations

We calculate a three dimensional arithmetic European call option by a Monte Carlo (MC) simulation and compare the results with the approximations in Proposition 3.2 - 3.5. Table 1 shows that in the chosen constellation the Johnson approximation leads to the best results. This is probably the result from the additional moments. Furthermore, it strengthens the assumption in Remark 3.6 of zero probability on negative values after matching $B(T)$ to the Type II distribution of the Johnson family.

In Figure 1 we calculate a second example, where the relative error of the approximated prices with respect to the prices from a MC simulation of a five dimensional problem are

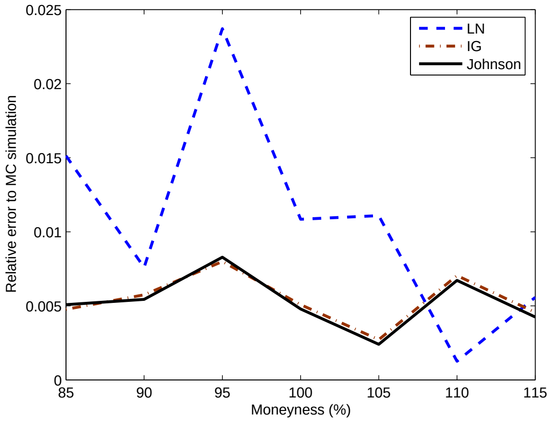

Figure 1: This figure shows the relative errors of the closed-form approximations with respect to the five dimensional MC simulation. The parameters are chosen as $t = 0$, $T = 1$, $\mathbf{S}(0) = (15, 30, 25, 20, 10)'$, $\boldsymbol{\sigma} = (0.20, 0.40, 0.30, 0.10, 0.25)'$, $\mathbf{w} = (1, 1, 1, 1, 1)'$ and $\mathbf{r} = (0.02, 0.04, 0.06, 0.10, 0.05)'$. The time interval is split into 250 subintervals. The MC simulation is the mean and the standard deviation of 10 simulations with $100'000$ different paths. The moneyness is defined by $K/B(0)$ where K is the strike.

shown. The inverse-gamma approximation has nearly everywhere the same relative error than the Johnson approximation. In fact, for higher dimension the inverse-gamma should lead to better results than the Johnson approximation because of the limiting behavior of sums of correlated lognormal distributed random variables. The correlation structure of the five dimensional basket is given by

$$\varrho = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.6 & 0.0 \\ 0.5 & 1 & 0.5 & 0.3 & 0.7 \\ 0.5 & 0.5 & 1 & 0.1 & 0.5 \\ 0.6 & 0.3 & 0.1 & 1 & 0.2 \\ 0.0 & 0.7 & 0.5 & 0.2 & 1 \end{pmatrix}.$$

3.3 A Numerical Approach with Finite Differences

It is well-known that the one dimensional Black-Scholes PDE can be solved with finite differences or finite elements. One obtains the price for different initial values $S(0)$. We find an approach of pricing multi-asset options with finite differences on meshfree radial basis functions (RBF) in [Fasshauer et al., 2004]. The paper presents an elegant way of approximating American style basket options. Fortunately, one can use the same procedure for pricing European options as well as exotic options like knock-outs. The large deficit is the exponential increasing number of calculations for higher dimensions. We present here the basic idea and show numerical calculations for a two dimensional basket. The Black-Scholes PDE for higher dimension is given by

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} \varrho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r-d_i)S_i\frac{\partial P}{\partial S_i} - rP = 0 \qquad \text{in } \Omega, \quad 0 \leq t \leq T. \tag{3.6}$$

Here, r denotes the domestic and d_i the foreign risk-free rate (respectively the negative dividend yield in case of equity underlying assets). For an European put the boundary conditions are given by

$$P(\mathbf{S},T) = F(\mathbf{S}) \qquad \text{in } \Omega, \tag{3.7}$$

$$P(\mathbf{S},t) = g_i(\mathbf{S},t) \qquad \text{in } \Omega_i, \quad i = 1,\ldots,n, \tag{3.8}$$

$$\lim_{S_i \to \infty} P(\mathbf{S},t) = 0 \qquad \text{in } \Omega, \quad i = 1,\ldots,n, \tag{3.9}$$

where $F(\mathbf{S})$ denotes the payoff function given by

$$F(\mathbf{S}) = \left(K - \mathbf{w}'\mathbf{S}\right)^{+}.$$

The domain Ω is defined by $\Omega = \{\mathbf{S}|\mathbf{S} > \mathbf{0}\}$. Here, the domain Ω_i denotes the boundaries of Ω along which the price $S_i = 0$. Note, if one of the asset prices is zero at time t^*, then the asset will be worthless for all $t \geq t^*$. Thus, the functions g_i specifying the boundary conditions are the solutions of the associated $(n-1)$-dimensional Black-Scholes problem. We have to be aware that for American problems the PDE in (3.6) is only full-filled as long as the option is alive. We introduce the notation $\bar{\mathbf{S}}(t) = (\bar{S}_1(t), \ldots, \bar{S}_n(t))$ to represent the moving boundary. Hence, we get the following additional constraints in the American case:

$$
\begin{aligned}
S_i &> \bar{S}_i(t) && i = 1, \dots, n, \quad 0 \le t < T, \\
P\left(\bar{\mathbf{S}}(t), t\right) &= F\left(\bar{\mathbf{S}}(t)\right) && 0 \le t < T, \\
\frac{\partial P(\bar{\mathbf{S}}, t)}{\partial S_i} &= -w_i && i = 1, \dots, n, && (3.10) \\
P(\mathbf{S}, t) - F(\mathbf{S}) &\ge 0 && \text{in } \Omega. && (3.11)
\end{aligned}
$$

Condition (3.10) is known as the so-called smooth pasting condition. It leads to a smooth transition at $\bar{\mathbf{S}}$. Obviously, $\bar{\mathbf{S}}(t) = \mathbf{0}$ in the European case. Early exercise is permitted for American option, therefore the positivity constraint in (3.11) must hold. The approach of eliminating the moving boundary from the above formulation follows from [Fasshauer et al., 2004] and [Nielsen et al., 2002]. They add a penalty term to the Black-Scholes equation in (3.6) and thereby convert the problem to one on a fixed domain. The penalty term is chosen such that the option price stays above the payoff function for all times $t \in [0, T]$. The penalty term is given by

$$
\frac{\epsilon C}{P + \epsilon - q}, \tag{3.12}
$$

where $0 < \epsilon \ll 1$ is a small regularization parameter, $C > rK$ is a positive constant and

$$
q(\mathbf{S}) = K - \mathbf{w}'\mathbf{S}
$$

is the barrier function. Note, the penalty term in (3.12) is of order ϵ if $P \gg q$, and it increases towards C as $P \to q$. Hence, as long as the option price is far above the moving boundary the problem is reduced to the European case given by (3.6) - (3.9). Otherwise, the constant C is large enough, such that

$$
P(\mathbf{S}, t) \ge \max\left(q(\mathbf{S}), 0\right)
$$

for all $t \in [0, T]$, which is identical to the condition in (3.11). It turns out that the mild assumption of $C > rK$ suffices to hold positivity constraint in case of an implicit scheme for the penalty term. This is shown in [Nielsen et al., 2002]. In case of an explicit scheme we need another condition for the size of the time step Δt in the numerical method of Section 3.3.1. Since we do not consider an explicit scheme we do not face this problem here. Finally, we obtain a parabolic nonlinear PDE by adding the penalty term (3.12) to the multi-asset Black-Scholes equation (3.6):

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} \varrho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r-d_i)S_i\frac{\partial P}{\partial S_i} - rP + \frac{\epsilon C}{P+\epsilon-q} = 0 \qquad \text{in } \Omega, \quad 0 \le t \le T.$$

The terminal and boundary conditions are reduced to the conditions in (3.7) - (3.9).

Remark 3.7 *By adding additional boundary conditions, we can solve exotic problems like knock-out options. We show one example in Section 3.3.2. European options are special cases of the above problem. By choosing $C = 0$ we get rid of the penalty term and obtain therefore the European problem.*

3.3.1 Meshfree Numerical Approximation Method

The meshfree RBF approach to the solution of parabolic PDEs is similar to the spectral method of lines approach. We assume that the value P corresponding to the asset prices $\mathbf{S}$ and time t can be expanded in the form

$$P(\mathbf{S}, t) = \sum_{i=1}^{N} a_i(t)\phi(\|\mathbf{S}-\mathbf{x}_i\|), \tag{3.13}$$

where time and space have been decoupled. The radial function ϕ determines the approximation space as the span of the functions $\phi(\|\cdot - \mathbf{x}_1\|), \ldots, \phi(\|\cdot - \mathbf{x}_N\|)$. The centers $\mathbf{x}_i$, $i = 1, \ldots, N$, form a discretization of the domain Ω. For our numerical calculations we use Gaussian RBF, given by

$$\phi(\|\mathbf{S}-\mathbf{x}_i\|) = e^{-\|\mathbf{S}-\mathbf{x}\|^2/c^2},$$

with a user-selectable shape parameter c. The function ϕ is dimension-blind since there is only the norm inside. The norm $\|\cdot\|$ is the standard Euclidean norm. To derive a discrete system we have to calculate partial derivatives of P. By differentiating (3.13), we obtain

$$\begin{aligned}\frac{\partial P}{\partial t} &= \sum_{i=1}^{N} \frac{\partial a_i(t)}{\partial t} \phi(\|\mathbf{S}-\mathbf{x}_i\|),\\ \frac{\partial P}{\partial S_k} &= \sum_{i=1}^{N} a_i(t) \frac{\partial \phi(\|\mathbf{S}-\mathbf{x}_i\|)}{\partial S_k},\\ \frac{\partial^2 P}{\partial S_k \partial S_l} &= \sum_{i=1}^{N} a_i(t) \frac{\partial^2 \phi(\|\mathbf{S}-\mathbf{x}_i\|)}{\partial S_k \partial S_l},\end{aligned}$$

where the partial derivatives of ϕ are given by

$$\begin{aligned}\frac{\partial \phi(\|\mathbf{S}-\mathbf{x}_i\|)}{\partial S_k} &= -\frac{2(S_k - x_{i,k})}{c^2} e^{-\|\mathbf{S}-\mathbf{x}_i\|^2/c^2},\\ \frac{\partial^2 \phi(\|\mathbf{S}-\mathbf{x}_i\|)}{\partial S_k \partial S_l} &= -\frac{4(S_k - x_{i,k})(S_l - x_{i,l})}{c^4} e^{-\|\mathbf{S}-\mathbf{x}_i\|^2/c^2} \qquad \text{for } k \neq l,\\ \frac{\partial^2 \phi(\|\mathbf{S}-\mathbf{x}_i\|)}{\partial S_k^2} &= -\frac{4(S_k - x_{i,k})^2 - 2c^2}{c^4} e^{-\|\mathbf{S}-\mathbf{x}_i\|^2/c^2},\end{aligned}$$

with $x_{i,k}$ denoting the k^{th} component of the center $\mathbf{x}_i$. Now we collocate at the centers $\mathbf{x}_i$, $i = 1, \ldots, N$, forming a discretization of the spatial part of the partial differential equation. This results in the system of nonlinear ODEs for $\mathbf{a} = (a_1, \ldots, a_N)'$:

$$\Phi \frac{\partial \mathbf{a}}{\partial t} + R\mathbf{a} + Q(\mathbf{a}) = \mathbf{0}.$$

Here,

$$R = \frac{1}{2} \sum_{k,l}^{n} \varrho_{kl} \sigma_k \sigma l \Phi_{\mathbf{S}}^{(k,l)} + \sum_{k=1}^{n} (r - d_k) \Phi_{\mathbf{S}}^{(k)} - r\Phi,$$

with the matrix Φ determined by $\Phi_{ij} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)$ and $\Phi_{\mathbf{S}}^{(k)}$ and $\Phi_{\mathbf{S}}^{(k,l)}$ given by

$$\begin{aligned}\Phi_{\mathbf{S},ij}^{(k)} &= \mathbf{x}_{i,k} \left. \frac{\partial \phi(\|\mathbf{S}-\mathbf{x}_j\|)}{\partial S_k} \right|_{\mathbf{S}=\mathbf{x}_i},\\ \Phi_{\mathbf{S},ij}^{(k,l)} &= \mathbf{x}_{i,k} \mathbf{x}_{i,l} \left. \frac{\partial^2 \phi(\|\mathbf{S}-\mathbf{x}_j\|)}{\partial S_k \partial S_l} \right|_{\mathbf{S}=\mathbf{x}_i}.\end{aligned}$$

The components of the vector $Q(\mathbf{a})$ are given by

$$Q_i(\mathbf{a}) = \frac{\epsilon C}{\Phi_i \mathbf{a} + \epsilon - q(\mathbf{x}_i)}, \quad i = 1, \ldots, N,$$

with Φ_i denoting the i^{th} row of the matrix Φ. In order to resolve the time component, we use a θ-method, i.e.,

$$\Phi \frac{\mathbf{a}^{n+1} - \mathbf{a}^n}{\Delta t} + \theta R \mathbf{a}^{n+1} + (1-\theta) R \mathbf{a}^n + \theta Q(\mathbf{a}^{n+1}) + (1-\theta) Q(\mathbf{a}^n) = \mathbf{0}, \tag{3.14}$$

where $\mathbf{a}^n = \mathbf{a}(n\Delta t)$ with Δt the time step chosen for the discretization of the time interval. To solve (3.14) with standard algorithms for linear systems we use an implicit method by replacing $\mathbf{a}^n$ in the penalty term by $\mathbf{a}^{n+1}$. Such implicit methods are well studied and stable and furthermore, we do not have a constraint on Δt as mentioned above. (See [Fasshauer et al., 2004] for further details.) Then the implicit (linear) version of (3.14) is given by

$$(\Phi - (1-\theta)\Delta t R)\, \mathbf{a}^n = (\Phi + \theta \Delta t R)\, \mathbf{a}^{n+1} + \Delta t Q(\mathbf{a}^{n+1}),$$

where $0 \leq \theta \leq 1$. Typically we choose $\theta = 0.5$ which is the Crank-Nicolson method with larger stability than for other values of θ. The boundary conditions in (3.8) are given as the solution of the $(n-1)$-dimensional penalized Black-Scholes equation. Therefore it is an iterative procedure from dimension n back to dimension 1, where for the single asset problem g_i is the solution of

$$\begin{aligned} \frac{\partial g_i}{\partial t} + \frac{1}{2}\sigma_i^2 S_i^2 \frac{\partial^2 g_i}{\partial S_i^2} + (r - d_i) S_i \frac{\partial g_i}{\partial S_i} - r g_i + \frac{\epsilon C}{g_i + \epsilon - q(S_i)} &= 0 && 0 \leq S_i,\ 0 \leq t \leq T, \\ g_i(S_i, T) &= (K - w_i S_i)^+ && 0 \leq S_i, \\ g_i(0, t) &= K && 0 \leq t \leq T, \\ \lim_{S_i \to \infty} g_i(S_i, t) &= 0. \end{aligned}$$

Here, $q(S_i) = K - w(i)S(i)$.

3.3.2 Numerical Results

We calculate according to the following algorithm solutions for a two dimensional problem. Higher dimensions are not complicated to implement, but the number of calculations is increasing exponentially in the dimension n such that we are not able to solve higher dimensional basket options with the given software. The algorithm we use looks like:

1. Choose a time step Δt and a value of $0 \leq \theta \leq 1$,
2. assemble the matrices Φ and R,
3. compute the matrices $R_1 = \Phi - (1-\theta)\Delta t R$ and $R_2 = \Phi + \theta \Delta t R$,
4. initialize the solution vector $\mathbf{P}$ via $P_i = (K - \mathbf{w}'\mathbf{x}_i)^+$, $i = 1, \ldots, N$.
5. For each time step:
   a) Update the coefficients $\mathbf{a}$ by solving $\Phi\mathbf{a} = \mathbf{P}$,
   b) compute $\mathbf{b} = R_2\mathbf{a}$ and the vector $Q(\mathbf{a})$,
   c) find the next coefficients $\mathbf{a}$ by solving the linear system $R_1\mathbf{a} = \mathbf{b} + \Delta t Q(\mathbf{a})$,
   d) update the solution vector $\mathbf{P} = \Phi\mathbf{a}$,
   e) and finally enforce the boundary conditions by backward iteration in the dimension.

Enforcing boundary conditions is not always easy to implement. In the one dimensional problem we only have $P_1 = 0$ and $P_N = K$. In the two dimensional European case one can use the standard Black-Scholes formula on the boundaries such that one gets rid of the backward iteration. In the American case we have to solve the 1-dimensional algorithm on the boundaries. Figure 2 shows the case of an European put option. Furthermore, it compares the results with the inverse gamma approximation from Proposition 3.4. In the OTM (out-of-the-money) area the error is practically zero. It increases on the boundaries and at the ATM (at-the-money) line. This follows from the discontinuity of the first derivative of the payoff function and from the implementation of the boundaries. But in general it proofs the

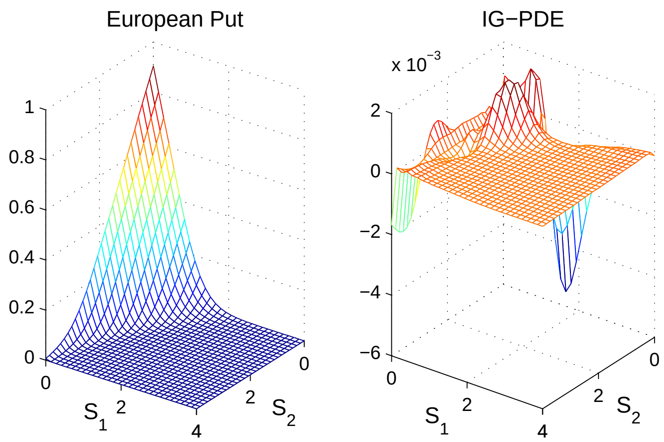

Figure 2: An European put option calculated with finite differences and standard Black-Scholes formula on the boundaries. The parameters are $\mathbf{w} = (0.3, 0.7)'$, $\varrho_{12} = -0.5$, $\boldsymbol{\sigma} = (0.2, 0.3)'$, $r = 0.1$, $\mathbf{d} = (0, 0)'$ and strike $K = 1$. The meshfree approximation is based on 30×30 grid points on $\Omega = [0, 4] \times [0, 4]$ and $\theta = 0.5$. The time runs from $t_0 = 0$ to $T = 1$ and is divided in 100 equal time steps. The right figure is the difference between the finite differences method and the IG approximation from Proposition 3.4.

goodness of the inverse gamma approximation although we only have two underlying assets. The American case in Figure 3 uses the same parameters than the European example. We see that the positivity constraint is fulfilled since the time-value is larger than zero. Figure 4 shows an European and an American knock-out put option. One remarks that the price of the American style option is much more higher than for the European option although the parameters are the same. The difference is the early exercise premium. Furthermore, one sees that the descent at the knock-out line is much steeper in the American case. Basically, we should have a vertical wall at the knock-out line in the American case, which is not the case and follows from the penalty term. The penalty term leads only to a good approximation and not to the exact value.

4 Hedging of Basket Options

In order to apply dynamic hedging strategies we have to find the Greeks of basket options. We find simple expressions in the Black-Scholes model for one underlying asset. It becomes more complicated in the case of a multidimensional basket option. A simple way of calcu-

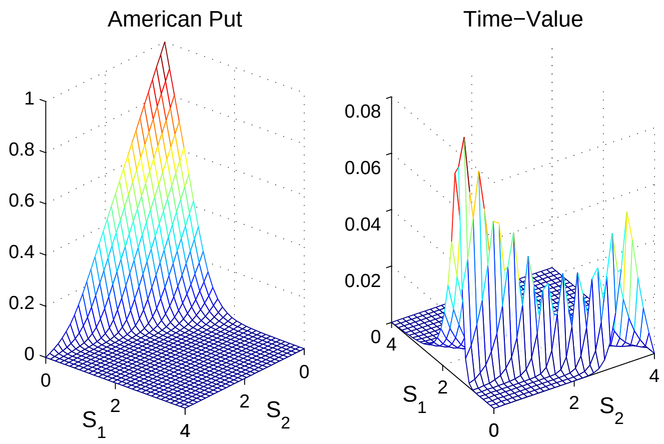

Figure 3: An American put option calculated with the algorithm given in Section 3.3.2. The parameters are $\mathbf{w} = (0.3, 0.7)'$, $\varrho_{12} = -0.5$, $\boldsymbol{\sigma} = (0.2, 0.3)'$, $r = 0.1$, $\mathbf{d} = (0, 0)'$ and strike $K = 1$. The meshfree approximation is based on 30×30 grid points on $\Omega = [0, 4] \times [0, 4]$ and $\theta = 0.5$. The parameters for the penalty term are $C = rK$ and $\epsilon = 0.01$. The time runs from $t_0 = 0$ to $T = 1$ and is divided in 100 equal time steps. The right figure is the time-value which has to be positive at each node for the American problem.

lating Greeks numerically is by approximating the differential quotient by finite differences. Especially in the case of closed-form approximations we can use this approach since the solutions are smooth enough such that the finite difference approximation converges in the limit to the derivatives.

First, we consider the geometric case and afterwards we try to find Greeks for the arithmetic problem. Additionally, we devote a section to static hedging. We derive a super-replicating portfolio of an arithmetic basket option and give also a sub-replicating strategy.

4.1 Geometric Basket

One finds the Greeks of a geometric basket option by applying the chain rule to (3.3). The results for *delta* and *vega* are given in the following proposition. One can calculate in the same manner further Greeks.

Proposition 4.1 *Assuming the price of an European geometric basket option C is given by*

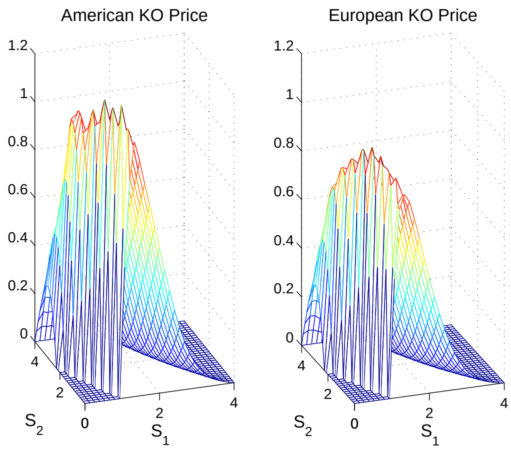

Figure 4: On the left side we see an American knock-out put option and on the right side an European knock-out put option. Both are calculated with the algorithm given in Section 3.3.2. The parameters are $\mathbf{w} = (0.3, 0.7)'$, $\varrho_{12} = -0.5$, $\boldsymbol{\sigma} = (0.2, 0.3)'$, $r = 0.1$, $\mathbf{d} = (0.04, 0.01)'$ and strike $K = 2$. The meshfree approximation is based on 30×30 grid points on $\Omega = [0, 4] \times [0, 4]$ and $\theta = 0.5$. The parameters for the penalty term in the American case are $C = rK$ and $\epsilon = 0.01$. The time runs from $t_0 = 0$ to $T = 1$ and is divided in 100 equal time steps. The knock-out level is chosen as 0.7.

(3.2), then

$$
\begin{aligned}
vega_k = \frac{\partial C}{\partial \sigma_k} = \frac{B d_2 \Phi(d_2)}{\sigma} & \left(\left(\frac{d_2}{\sigma} - 2\sqrt{T} \right) a_k \sum_{i=1}^{n} a_i \varrho_{ik} \sigma_i + a_k \sigma_k \sqrt{T} \right) \\
& - \frac{K e^{-rT} d_1 \Phi(d_1)}{\sigma} \left(\frac{d_1 a_k}{\sigma} \sum_{i=1}^{n} a_i \varrho_{ik} \sigma_i + a_k \sigma_k \sqrt{T} \right), \\
\Delta_k = \frac{\partial C}{\partial S_k} = \frac{a_k B}{S_k} & \left(\Phi(d_2) \left(1 - \frac{d_2}{\sigma\sqrt{T}} \right) + \Phi(d_1) e^{-rT} \frac{K d_1}{B \sigma \sqrt{T}} \right),
\end{aligned}
$$

for $k \in \{1, \ldots, n\}$.

Proof: The results in the proposition can be derived by applying the chain rule to C.

$$
\begin{aligned}
\frac{\partial C}{\partial \sigma_k} &= \frac{\partial C}{\partial \sigma} \frac{\partial \sigma}{\partial \sigma_k} + \frac{\partial C}{\partial a} \frac{\partial a}{\partial \sigma_k}, \\
\frac{\partial C}{\partial S_k} &= \frac{\partial C}{\partial B} \frac{\partial B}{\partial S_k}.
\end{aligned}
$$

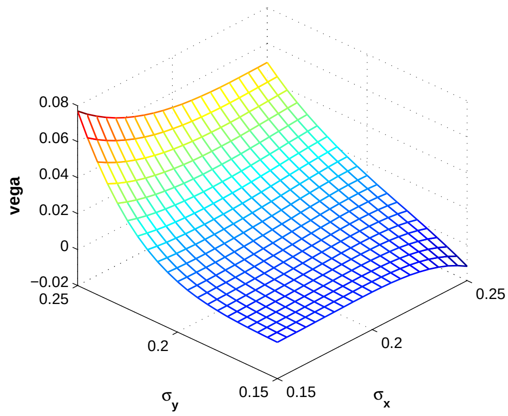

Figure 5: $Vega$ of a two factor geometric basket with varying volatility on the underlying assets. The parameters are $\mathbf{r} = (0.05, 0.05)'$, $\varrho = -0.7$, $\mathbf{a} = (0.5, 0.5)'$, $\mathbf{S}(0) = (30, 30)'$ and the strike $K = 40$. The time runs from 0 to 1. It does not matter whether we look at *vega* with respect to σ_x or *vega* with respect to σ_y since the basket is symmetric.

The single terms are straightforward to calculate. One obtains

$$\begin{aligned}
\frac{\partial C}{\partial \sigma} &= B d_2 \Phi(d_2)\left(\frac{d_2}{\sigma} - 2\sqrt{T}\right) - K e^{-rT} d_1 \Phi(d_1)\frac{d_1}{\sigma}, \\
\frac{\partial \sigma}{\partial \sigma_k} &= \frac{a_k}{\sigma}\sum_{i=1}^{n} a_i \varrho_{ik}\sigma_i, \\
\frac{\partial C}{\partial a} &= -B\Phi(d_2)\frac{d_2\sqrt{T}}{\sigma} + e^{-rT} K \Phi(d_1)\frac{d_1\sqrt{T}}{\sigma}, \\
\frac{\partial a}{\partial \sigma_k} &= -a_k \sigma_k, \\
\frac{\partial C}{\partial B} &= \Phi(d_2)\left(1 - \frac{d_2}{\sigma\sqrt{T}}\right) + e^{-rT}\frac{K}{B}\Phi(d_1)\frac{d_1}{\sigma\sqrt{T}}, \\
\frac{\partial B}{\partial S_k} &= \frac{a_k B}{S_k}
\end{aligned}$$

for $k \in \{1, \ldots, n\}$, from where the results follow. □

In Figure 5 one sees $vega_x$ respectively $vega_y$ of a geometric basket option which is symmetric in the underlying assets. Interestingly, we obtain negative *vega* for certain values of $\boldsymbol{\sigma}$.

4.2 Arithmetic Basket with the Closed-Form Approximations

A closed-form derivation of the Greeks is in some cases possible but complicated to obtain. In the case of approximating via an inverse gamma or Johnson approach, see Proposition 3.4 and 3.5, the calculation of $\Delta_k := \dfrac{\partial C}{\partial S_k}$, $k \in \{1, \dots, n\}$, are probably not solvable in closed-form. Therefore, it is in most cases useful to calculate the Greeks numerically by approximating the differential quotient by finite differences. Anyway, we want to derive some closed-form solutions for the Greeks of the lognormal approximation. In a second part, we look at the possibility of static hedging. In many examples of one underlying asset, also for exotic options, it is possible to find static replicating portfolios. These paradigms are described for example in [Carr et al., 1998] and [Derman et al., 1995]. In the context of arithmetic basket options we present a super- and sub-replicating strategy.

4.2.1 Greeks of the Lognormal Approximation

From the lognormal approximation of Proposition 3.2 we can derive Δ_k and $vega_k := \dfrac{\partial C}{\partial \sigma_k}$ for $k \in \{1, \dots, n\}$ in closed-form. We need first a remark for the calculation.

Remark 4.2 *The price in Proposition 3.2 is given as*

$$C = e^{-rT} \left[F\Phi(d_1) - K\Phi(d_2) \right],$$

where $d_1 = \frac{\log(F/K)+v/2}{\sqrt{v}}$ *and* $d_2 = d_1 - \sqrt{v}$. $\Phi(.)$ *denotes the standard normal CDF. Trivial calculations show the following identities:*

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F},$$
$$\frac{\partial d_1}{\partial v} = \frac{\partial d_2}{\partial v} + \frac{1}{2\sqrt{v}}.$$

Furthermore, we have the following lemma.

Lemma 4.3 *With the variables chosen as above, we obtain*

$$F\Phi'(d_1) - K\Phi'(d_2) = 0.$$

Proof: Note, $(d_1 + d_2)(d_1 - d_2) = 2\log(F/K)$. Then it is straightforward to see that

$$\begin{aligned}\frac{\Phi'(d_1)}{\Phi'(d_2)} &= \exp\left(-\frac{1}{2}(d_1^2 - d_2^2)\right)\\ &= \exp\left(-\frac{1}{2}(d_1 + d_2)(d_1 - d_2)\right)\\ &= \exp\left(-\log(F/K)\right) = \frac{K}{F},\end{aligned}$$

from where the lemma follows. □

We use this lemma to calculate Δ_k and $vega_k$ and obtain the next proposition.

Proposition 4.4 *Let the price of an European basket call option be given by* $C_{call,LN}$. *Then,* Δ_k *and* $vega_k$ *are*

$$\begin{aligned}\Delta_k &= \frac{\partial C_{call,LN}}{\partial S_k} = e^{-rT} w_k e^{r_k T}\left(\Phi(d_1) + \frac{\Phi'(d_1)}{\sqrt{v}}\left(\frac{1}{FM_2^*}\sum_{i=1}^n w_i F_i e^{\varrho_{ik}\sigma_i\sigma_k T} - 1\right)\right),\\ vega_k &= \frac{\partial C_{call,LN}}{\partial \sigma_k} = e^{-rT}\frac{\Phi'(d_1)}{2\sqrt{v}FM_2^*}\sum_{i,k=1}^n w_i w_k F_i F_k \varrho_{ik}\sigma_k T e^{\varrho_{ik}\sigma_i\sigma_k T},\end{aligned}$$

for $k \in \{1, \dots, n\}$.

Proof: We use the chain rule for the derivation of $C = C_{call,LN}$.

$$\frac{\partial C}{\partial S_k} = \frac{\partial C}{\partial F}\frac{\partial F}{\partial S_k} + \frac{\partial C}{\partial v}\frac{\partial v}{\partial S_k}.$$

Then, we calculate the single expressions and obtain

$$
\begin{aligned}
\frac{\partial C}{\partial F} &= e^{-rT}\left(\Phi(d_1) + F\Phi'(d_1)\frac{\partial d_1}{\partial F} - K\Phi'(d_2)\frac{\partial d_2}{\partial F}\right) \\
&= e^{-rT}\left(\Phi(d_1) + \frac{\partial d_1}{\partial F}(F\Phi'(d_1) - K\Phi'(d_2))\right) = e^{-rT}\Phi(d_1), \\
\frac{\partial F}{\partial S_k} &= w_k e^{r_k T}, \\
\frac{\partial C}{\partial v} &= e^{-rT}\left(F\Phi'(d_1)\frac{\partial d_1}{\partial v} - K\Phi'(d_2)\frac{\partial d_2}{\partial v}\right) \\
&= e^{-rT} F \frac{\Phi'(d_1)}{2\sqrt{v}}, \\
\frac{\partial v}{\partial S_k} &= \frac{\partial \log(M_2^*)}{\partial S_k} = \frac{1}{M_2^*}\frac{\partial M_2^*}{\partial S_k} \\
&= \frac{2 w_k e^{r_k T}}{F}\left(\frac{1}{F M_2^*}\sum_{i=1}^{n} w_i F_i e^{\varrho_{ik}\sigma_i\sigma_k T} - 1\right).
\end{aligned}
$$

Putting everything together leads us to the first result. The second result is similar:

$$
\begin{aligned}
\frac{\partial C}{\partial \sigma_k} &= \frac{\partial C}{\partial v}\frac{\partial v}{\partial \sigma_k}, \\
\frac{\partial v}{\partial \sigma_k} &= \frac{1}{M_2^* F^2}\sum_{i,k=1}^{n} w_i w_k F_i F_k \varrho_{ik}\sigma_i T e^{\varrho_{ik}\sigma_i\sigma_k T},
\end{aligned}
$$

and the proof follows. □

We can calculate further Greeks by applying the chain rule. Unfortunately, the expressions become quite fast complicated and therefore we omit here the calculations for other Greeks. One can derive results numerically by approximating the differential quotient by finite differences. E.g. for Γ_{kl}, we have

$$
\Gamma_{kl} = \frac{\partial C_{call,LN}}{\partial S_k \partial S_l} = \frac{\partial \Delta_k}{\partial S_l} \approx \frac{\Delta_k(S_1,\dots,S_l+\Delta S_l,\dots,S_n) - \Delta_k(S_1,\dots,S_l,\dots,S_n)}{\Delta S_l},
$$

for $k, l \in \{1,\dots,n\}$.

4.2.2 A Static Super-Replicating Strategy

The objective of this method is to find a super-replicating portfolio whose final payoff is always larger than the payoff of a basket call. We obtain, by using Jensen's inequality for the final payoff,

$$\begin{aligned} C(T) = \left(\sum_{i=1}^{n} w_i S_i(T) - K\right)^+ &= \left(\sum_{i=1}^{n} w_i \left(S_i(T) - \frac{b_i}{w_i}K\right)\right)^+ \\ &\leq \sum_{i=1}^{n} w_i \left(S_i(T) - \frac{b_i}{w_i}K\right)^+, \end{aligned}$$

where $\sum_{i=1}^{n} b_i = 1$. Therefore, the payoff of the portfolio consisting of n plain vanilla call options is larger or equal to that of the corresponding basket call option. From a no-arbitrage argument we follow that the discounted expected final payoff under risk-neutral measure $\mathbb{Q}$ also fulfills the inequality

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{i=1}^{n} w_i S_i(T) - K\right)^+\right] \leq \sum_{i=1}^{n} w_i e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(S_i(T) - \frac{b_i}{w_i}K\right)^+\right]. \tag{4.1}$$

For the purpose of hedging we would like to look for plain vanilla call options with strikes depending on b_i such that cost of the super-replicating portfolio is minimized. Hence, we have to solve the following optimization problem:

$$\begin{aligned} &\min_{b_1,\dots,b_n} \sum_{i=1}^{n} w_i e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(S_i(T) - \frac{b_i}{w_i}K\right)^+\right] \\ &s.t. \quad \sum_{i=1}^{n} b_i = 1. \end{aligned}$$

The solution of this optimization problem is given in the next proposition.

Proposition 4.5 *Suppose the underlying assets of the basket $B(t)$ follow geometric Brownian motions and the Black-Scholes model is valid, then the optimal b_i^*, $i = 1, ..., n$, satisfying (4.1),* *are uniquely obtained by solving a set of non-linear equations:*

$$b_i = \frac{w_i S_i}{K}\left(\frac{b_1 K}{w_1 S_1}\right)^{\sigma_i/\sigma_1} \exp\left(T\left(1-\frac{\sigma_i}{\sigma_1}\right)\left(r_i + \frac{1}{2}\sigma_1\sigma_i\right)\right) \qquad i = 1,..,n,$$

$$\sum_{i=1}^{n} b_i = 1.$$

Proof: See [Su, 2005]. □

Not all optimal strikes $k_i^{opt} = \frac{b_i^*}{w_i}$ are traded in the market. Let $\kappa_i = (k_i^0, ..., k_i^m)$ denote the set of traded strikes of underlying S_i in increasing order and $k_i^0 = 0$. Since the Black-Scholes price is convex in the strike, we have

$$C_i\left(k_i^{opt}\right) \leq \beta^* C_i(k_i^j) + (1-\beta^*)\, C_i\left(k_i^{j+1}\right),$$

where k_i^j and k_i^{j+1} are the neighboring strikes of k_i^{opt}, i.e. $k_i^j < k_i^{opt} < k_i^{j+1}$, and $\beta^* = \frac{k_i^{j+1} - k_i^{opt}}{k_i^{j+1} - k_i^j}$. Hence, the upper bound for a basket call can be expressed as

$$\sum_{k_i^{opt} traded} w_i e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(S_i(T) - \frac{b_i^*}{w_i}K\right)^+\right] +$$
$$\sum_{k_i^{opt} non-traded} w_i e^{-rT}\left(\beta^*\mathbb{E}^{\mathbb{Q}}\left[\left(S_i(T) - k_i^j\right)^+\right] + (1-\beta^*)\,\mathbb{E}^{\mathbb{Q}}\left[\left(S_i(T) - k_i^{j+1}\right)^+\right]\right).$$

Considering the difference between the replicating portfolio of plain vanillas and the basket, one remarks that the difference is small for deep in-the-money options. This is not surprising since $(\sum_{i=1}^n w_i S_i(T) - K)^+ \approx \sum_{i=1}^n w_i S_i(T) - K$. Therefore, the inequality in (4.1) becomes approximately an equality. A similar argument holds for deep OTM options since the basket options is near to zero. In Figure 6 we see exactly these results of limiting behavior. The correlation structure used in the example is

$$\varrho = \begin{pmatrix} 1 & 0.5 & -0.5 \\ 0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{pmatrix}.$$

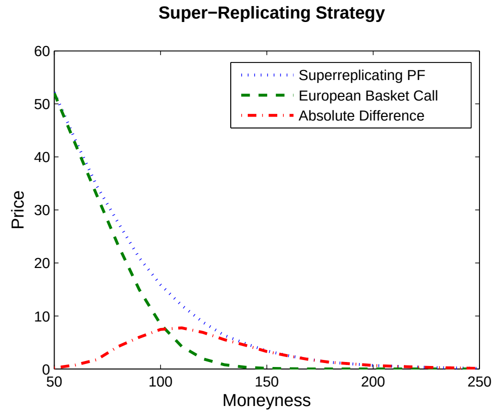

Figure 6: We compare the European basket call price for different moneyness with the super-replicating portfolio calculated with the results of Proposition 4.5. The parameters are $\mathbf{w} = (1, 1, 1)'$, $\boldsymbol{\sigma} = (0.40, 0.30, 0.25)'$, $\mathbf{r} = (0.03, 0.04, 0.05)'$, $\mathbf{d} = (0.04, 0.01)'$ and $\mathbf{S}(0) = (35, 25, 40)$. The moneyness is defined as $K/B(0)$ and the maturity is at $T = 1$.

4.2.3 A Static Sub-Replicating Strategy

The above super-replicating strategy is a way to over-hedge a basket. From an economic viewpoint it is an expensive and not cost-efficient way of hedging. We are looking for a strategy which eliminates most of the risk and which is cheap. For the two asset case, $n = 2$, there are several papers based on copulas (see e.g. [Hobson et al., 2005]) which derive sharp sub-replicating portfolios. But in most cases we are dealing with more than two underlying assets.

Proposition 4.6 *A lower bound for all $n \geq 2$ is given by*

$$L = \sum_{i=1}^{n} w_i S_i(0) - e^{-rT} K,$$

i.e. price of a forward contract with strike K on the basket.

Proof: This lower bound follows from

$$\begin{aligned}\mathbb{E}\left[(\mathbf{w}'\mathbf{S}(T)-K)^+\right] &\geq \mathbb{E}\left[\mathbf{w}'\mathbf{S}-K\right]\\ &= e^{rT}\mathbf{w}'\mathbf{S}(0)-K=\mathbf{w}'\mathbf{F}-K.\end{aligned}$$

The proof follows by discounting with e^{-rT}. □

Since the final payoff of the lower bound L is smaller or equal the payoff of the basket call, we have a sub-replicating strategy.

5 Stochastic Correlation in an Arithmetic Basket Option

Most models assume that correlation is constant in time. Considering changes of historical correlation one remarks that they can be approximated by a mean-reverting process. Unfortunately, correlations are bounded in $[-1, 1]$. We have to find a process, which transforms the unbounded Brownian motion into boundaries. Two model, which fulfill this criteria, are

$$d\varrho = k(\bar{\varrho}-\varrho)dt + \lambda(1-\varrho^2)dW, \tag{5.1}$$

$$d\varrho = k(\bar{\varrho}-\varrho)dt + \lambda\sqrt{1-\varrho^2}dW. \tag{5.2}$$

Remark 5.1 *Dynamic (5.1) and (5.2) is only constrained to* $[-1.1]$ *in continuous time-step simulations. In discrete simulations it is possible to step outside the bounds.*

To overcome the problem of stepping outside $[-1, 1]$ in discrete simulations, we define another process by

$$\begin{aligned}dZ &= k(\bar{Z}-Z)dt + \sigma_Z dW, \\ \varrho &= \tanh(Z).\end{aligned} \tag{5.3}$$

The instantaneous correlation is the hyperbolic tangent of an Ornstein-Uhlenbeck process reverting to Z_0 with strength k. The correlation is constrained to $[-1, 1]$ also for finite-time step simulations. Note, in any case the volatility respectively the second moment of the basket increases with stochastic correlation. This follows directly by Jensen's inequality:

$$\begin{aligned}\mathbb{E}[B^2] &= \mathbb{E}\left[\mathbb{E}[B^2|\varrho]\right] = \mathbb{E}[M_2] = \mathbb{E}\left[\sum_{i,j=1}^{n} w_i w_j F_i F_j e^{(\varrho_{ij}\sigma_i\sigma_j T)}\right] \\ &= \sum_{i,j=1}^{n} w_i w_j F_i F_j \mathbb{E}\left[e^{(\varrho_{ij}\sigma_i\sigma_j T)}\right] \geq \sum_{i,j=1}^{n} w_i w_j F_i F_j e^{(\mathbb{E}[\varrho_{ij}]\sigma_i\sigma_j T)}\end{aligned}$$

The great advantage of (5.1) and (5.2) is the calibration challenge to historical data. The mean of a process with dynamics as in (5.1) and (5.2) is given by

$$\mathbb{E}[\varrho(t)] = e^{-kt}\varrho(0) + \bar{\varrho}\left(1 - e^{-kt}\right), \tag{5.4}$$

as in an Ornstein-Uhlenbeck process. We can calculate the second moment by an ODE. By using Itô's formula, one obtains

$$d(\varrho^2) = 2\varrho d\varrho + d < \varrho >, \tag{5.5}$$

where $< \cdot >$ denotes the quadratic variation. Let us denote by m_n the n^{th} moment of $\varrho(t)$. By taking expectation on both side of (5.5) and using Fubini's theorem, we obtain

$$\frac{dm_2}{dt} = 2k(\bar{\varrho}m_1 - m_2) + \lambda^2(1 - m_2).$$

This ODE can be solved in closed-form. Unfortunately, the expression becomes complicated and not useful for calibration. Therefore, we do some estimations for the variance, which leads us to much simpler expressions. Let us denote by $f(\varrho) := 1-\varrho^2$ as in (5.1) or $f(\varrho) := \sqrt{1-\varrho^2}$ as in (5.2). Then the variance of ϱ is

$$\begin{aligned}Var(\varrho(t)) &= \lambda^2 e^{-2kt}\int_0^t e^{2ks}\mathbb{E}\left[f^2(\varrho(s))\right] ds \\ &\leq \lambda^2 e^{-2kt}\int_0^t e^{2ks} ds = \frac{\lambda^2}{2k}\left(1 - e^{-2kt}\right).\end{aligned} \tag{5.6}$$

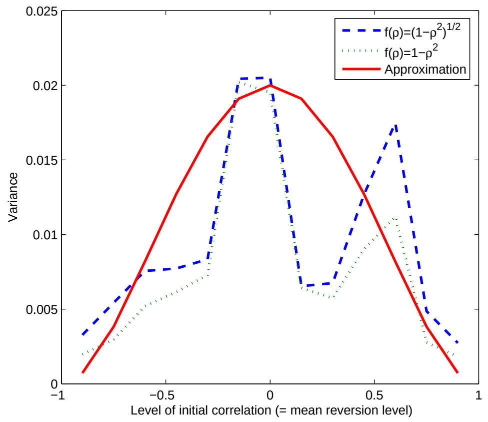

Figure 7: For each $\bar{\varrho}$ and $\varrho(0)$ chosen identical between $[-0.9, 0.9]$ we run a MC simulation of the stochastic correlation with $50'000$ different paths. The other parameters are $\lambda = 0.4$ and $k = 4$. The approximation is calculated with (5.7) and $f(x) = 1 - x^2$.

The upper bound follows from $f^2(\varrho) \leq 1$. An approximation of $\mathbb{E}\left[f^2(\varrho)\right]$ is $f^2(\mathbb{E}[\varrho])$, since we are dealing with a mean-reverting process. Hence, we have the following approximation for the variance of $\varrho(t)$:

$$Var(\varrho(t)) \approx \frac{\lambda^2 f^2(\mathbb{E}[\varrho])}{2k}\left(1 - e^{-2kt}\right). \tag{5.7}$$

A MC simulation of the variance of (5.1) and (5.2) is given in Figure 7. In the next section we are looking at the calibration of (5.1) and (5.2) to historical data. The process in (5.3) is difficult to calibrate.

5.1 Calibration to Data

In long-term consideration we can use the regression property of the dynamic in (5.2). We have the following non-linear regression

$$\varrho(t + \Delta t) = \varrho(t) + k(\bar{\varrho} - \varrho(t))\Delta t + \lambda\sqrt{1 - \varrho^2(t)}dW_t.$$

T	$\bar{\varrho}$			k			λ		
1	0.4686	3.1e-02	6.27%	9.3069	4.3e+00	86.14%	0.4078	7.8e-03	1.94%
2	0.4857	1.4e-02	2.87%	6.9813	2.0e+00	39.63%	0.4046	4.6e-03	1.15%
3	0.4910	9.0e-03	1.79%	6.3517	1.4e+00	27.03%	0.4041	4.1e-03	1.03%
4	0.4917	8.3e-03	1.65%	5.9875	9.9e-01	19.75%	0.4026	2.6e-03	0.65%
5	0.4942	5.8e-03	1.15%	5.7169	7.2e-01	14.34%	0.4005	4.5e-04	0.12%
10	0.4958	4.2e-03	0.85%	5.3313	3.3e-01	6.63%	0.4003	2.9e-04	0.07%
15	0.4987	1.3e-03	0.26%	5.2666	2.6e-01	5.33%	0.3998	2.4e-04	0.06%
20	0.4985	1.5e-03	0.31%	5.1759	1.7e-01	3.52%	0.3995	5.1e-04	0.13%
50	0.4992	7.7e-04	0.16%	5.0698	7.0e-02	1.40%	0.3991	9.4e-04	0.24%
75	0.4999	5.7e-05	0.01%	5.0503	5.0e-02	1.01%	0.3988	1.2e-03	0.30%
100	0.5001	1.4e-04	0.03%	5.0496	5.0e-02	1.00%	0.3989	1.1e-03	0.26%

Table 2: In each time step there are 1000 simulations of ϱ with the dynamic as in (5.2). The parameters are $\varrho(0) = 0.35$, $\bar{\varrho} = 0.5$, $k = 5$ and $\lambda = 0.4$. The columns show the estimated value, the absolute error and the relative error to the exact value.

Therefore, we get an expression which is iid standard normal by

$$X_t(\bar{\varrho}, k, \lambda) := \frac{\varrho(t + \Delta t) - \varrho(t) - k(\bar{\varrho} - \varrho(t))\Delta t}{\lambda\sqrt{(1 - \varrho^2(t))\Delta t}} \overset{iid}{\sim} N(0, 1).$$

A good approximation for $\bar{\varrho}$, at least in long-term considerations, is the mean of a given time-series. For the parameter k we use maximum likelihood estimates (MLE) on X_t, given by

$$k = \frac{\sum_t \frac{\varrho(t+\Delta t)-\varrho(t)}{1-\varrho^2(t)}(\bar{\varrho} - \varrho(t))}{\sum_t \frac{(\bar{\varrho}-\varrho(t))\Delta t}{1-\varrho^2(t)}(\bar{\varrho} - \varrho(t))}.$$

Finally, the volatility parameter λ can be estimated by the variance of the time-series and by the approximation (5.7). The MC simulation in Table 2 shows that the bias decreases in time.

Remark 5.2 *The approximations in Table 2 are an example with nice chosen parameters. For parameters like* $\bar{\varrho} \sim 1$ *we do not get such nice convergence behavior. Additionally, we have the problem of stepping out of* $[-1, 1]$*. In the discrete time simulation of this example we add an if-clause, which inverts the sign of* dW *if* $\varrho(t + \Delta t)$ *steps outside the bounds. Hence, we only cross the boundaries if the simulated value of* dW *is very large. This event has a very small probability. In fact we introduce an additional skewness. This seems not far from*

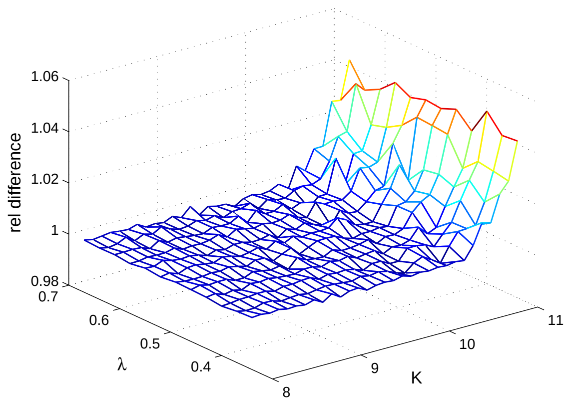

Figure 8: A MC simulation for the price of a two dimensional arithmetic basket with stochastic correlation relative to the price of a basket with constant correlation. The volatility λ of the correlation varies between 0.34 and 0.67. The strike goes from 8 to 11. The weights are chosen as $\mathbf{w} = (3, 1)'$. The initial values are from Table 3.

reality since, e.g., a correlation close to one moves with higher probability downwards than upwards. Large values of λ are another problem. We only have an approximation for λ in (5.7). From (5.6) we can calculate a lower bound for λ, which helps us in calibrating market data to the model.

5.2 Pricing an Arithmetic Basket with Stochastic Correlation

In Section 3.2 we see that there is no closed-form solution for the value of an arithmetic basket option. We derive approximations by matching the moments to different distributions. We would like to extend these approaches to stochastic correlations, but unfortunately for dimensions $n > 2$, we run into the problem of non positive semi-definite covariance matrices. Obviously, we do not face this problem for $n = 2$. But already for $n = 3$ the stochastic correlation must have an influence on the implied volatilities of the underlying assets. It is a question of time until we loose the positive semi-definiteness in a simulation by not changing the volatilities.

Nevertheless, we simulate a two dimensional arithmetic basket in Figure 8. The dynamic of the correlation is calibrated with historical $1M$ correlations from Sep'01 to Aug'06 of

EURUSD			USDSEK			ϱ			
σ	$S(0)$	$r_{dom} - r_{for}$	σ	$S(0)$	$r_{dom} - r_{for}$	$\bar{\varrho}$	$\varrho(0)$	k	λ_{cali}
7.83%	1.2783	-1.84%	9.93%	7.2223	2.12%	-86.02%	-95.98%	4.6	0.67

Table 3: Initial values on Aug 29, 2006 for the two dimensional example in Figure 8.

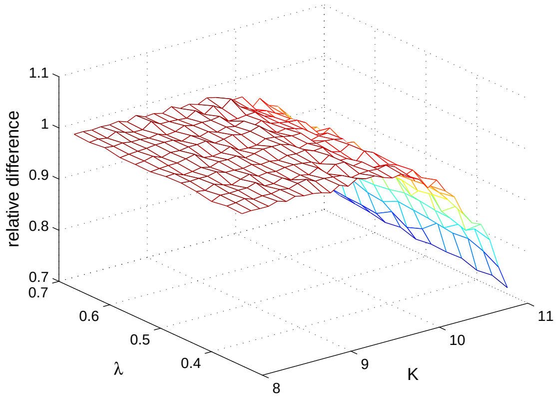

Figure 9: A MC simulation for the price of a two dimensional arithmetic basket with stochastic correlation relative to the price of a basket with constant correlation. The volatility λ of the correlation varies between 0.34 and 0.67. The strike goes from 8 to 11. The weights are chosen as $\mathbf{w} = (3, 1)'$. The initial values are from Table 3 but with $\varrho(0) = 0.5$.

{EURUSD, USDSEK}[2] returns. The initial values on Aug 29, 2006 can be found in Table 3. Note, from (5.6) we can derive the lower bound of λ. This lower bound is 0.34, which is the lower starting point of λ in Figure 8. We see that the price of the basket with stochastic correlation is basically the price with constant correlation. This is due to the current correlation $\varrho(0)$, which is used in the constant correlation basket, is not far from the long-term mean $\bar{\varrho}$. In Figure 9 we have the same parameters with the difference of $\varrho(0) = 0.5$. We see that the price can change up to 30% in the chosen interval for strike K.

5.3 Wishart Multi-Dimensional Stochastic Volatility

As remarked above it is impossible to simulate stochastic correlations without adjusting the volatilities to fulfill the constraint of semi-positive definiteness for more than two underlying

[2]EUR: Euro, USD: U.S. Dollar, SEK: Swedish Krona

assets. Wishart processes are a solution of this problem. They guarantee to have a positive semi-definite covariance (or correlation) structure. Let the multivariate dynamic of the underlying assets be given by

$$d\mathbf{S}_t = \mathrm{diag}(\mathbf{S}_t)\left(\mathbf{r}dt + \Sigma_t^{1/2} d\mathbf{W}\right), \tag{5.8}$$

where diag(·) is the diagonal matrix with the elements of $\mathbf{S}(t)$ on its diagonal, $\mathbf{r}$ denotes the risk-less returns, $\Sigma_t^{1/2}$ is the square root matrix of the covariance matrix and finally $\mathbf{W}_t$ is a n-dimensional Brownian motion with independent elements. Note, the square root of a positive semi-definite matrix exists and is also positive semi-definite. The dynamic of Σ_t is described by the following Wishart process

$$d\Sigma_t = \left(\Omega\Omega' + M\Sigma_t + \Sigma_t M'\right) dt + \Sigma_t^{1/2} dZ_t Q + Q' dZ_t' \Sigma_t^{1/2}, \tag{5.9}$$

where Ω, M and Q are $n \times n$-matrices and $Z(t)$ is a $n \times n$-dimensional matrix with independent standard Brownian motions as elements. M is assumed to be negative semi-definite. It drives the speed of mean-reversion. For completeness we write M and M' although the matrix is assumed to be negative semi-definite and therefore symmetric. Q drives the volatility of Σ. The main point of Wishart processes is the following proposition, which ensures that the problem is well-posed in the sense of a positive semi-definite covariance structure.

Proposition 5.3 *$\Sigma(t)$ is positive semi-definite for all t, if $\Omega\Omega' = \lambda Q'Q$ for some $\lambda \geq n-1$, where n is the dimension of the basket.*

Proof: See [Bru, 1991]. □

5.3.1 Pricing a Geometric Basket Option with Stochastic Covariance

Let us assume we have a geometric basket given by $B(t) = \prod_{i=1}^{n} S_i(t)$, where the single underlying assets follow the process given in (5.8). It is not difficult to see that the single underlying asset follows

$$S_i(t) = S_i(0) e^{\left(r_i - \frac{\sigma_{ii}^2}{2}\right)t + \sum_{j=1}^{n} \sigma_{ij} W_j},$$

where σ_{ij} denotes the entries of $\Sigma^{1/2}$. The geometric basket can therefore be written as

$$\begin{aligned} B_t &= B_0 \prod_{i=1}^n \exp\left\{ \left(r_i - \frac{\sigma_{ii}^2}{2} \right) t + \sum_{j=1}^n \sigma_{ij} W_j \right\} \\ &= B_0 \exp\left\{ \sum_{i=1}^n \left(r_i - \frac{\sigma_{ii}^2}{2} \right) t + \sum_{i,j=1}^n \sigma_{ij} W_j \right\} \\ &= B_0 \exp\left\{ \left(\mathbf{r}'\mathbf{e} - \frac{1}{2}\mathrm{Tr}(\Sigma_t) \right) t + \sum_{i,j=1}^n \sigma_{ij} W_j \right\}, \end{aligned}$$

where $\mathbf{e} = (1, \ldots, 1)'$. The basket dynamic follows by Itô's formula:

$$dB_t = B_t \left[\left(r - \frac{1}{2} \left(\mathrm{Tr}(\Sigma_t) - \mathbf{e}'\Sigma_t \mathbf{e} \right) \right) dt + \sum_{i,j=1}^n \sigma_{ij} dW_j \right].$$

The return r is defined as $r = \mathbf{r}'\mathbf{e}$. To derive the moments of the geometric basket we define $Y_t := \log B_t$ for which we have the following dynamic by applying Itô's formula:

$$dY_t = \left(r - \frac{1}{2}\mathrm{Tr}(\Sigma_t) \right) dt + \sum_{i,j=1}^n \sigma_{ij} dW_i. \tag{5.10}$$

The moments of B_t are given by the Laplace-transform of the $Y(t)$, since $\mathbb{E}[B_t^n] = \mathbb{E}[\exp(nY_t)]$. We follow a similar approach given in [Fonseca et al., 2005] to calculate the Laplace transform. They use the infinitesimal generator of the Wishart process Σ_t and of the tuple (Y_t, Σ_t) and apply the Feynman-Kač methodology.

Remark 5.4 *The infinitesimal generator is a concept from topology and describes actions on manifolds. We do not want to go into details, but we need it to use the Feynman-Kač argument. In our case, the infinitesimal generator is given by*

$$\mathcal{L}_{(Y,\Sigma)} f(Y, \Sigma) = \left. \frac{d}{dt} \right|_{t=0} f(Y, \Sigma),$$

where f is an enough regular function of Y and Σ.

Proposition 5.5 *The infinitesimal generator of the Wishart process* $\Sigma(t)$ *is given by*

$$\mathcal{L}_{\Sigma} = \mathrm{Tr}\left[(\Omega\Omega' + M\Sigma + \Sigma M')D + 2\Sigma D Q' Q D\right]$$

with

$$D = \left(\frac{\partial}{\partial \Sigma_{ij}}\right).$$

Proof: See [Bru, 1991]. □

Endowed with the previous result, we can find the infinitesimal generator of the tuple (Y_t, Σ_t).

Proposition 5.6 *Under the assumption that the Brownian motions* $\mathbf{W}(t)$ *and* $Z(t)$ *are independent, the infinitesimal generator of* (Y_t, Σ_t) *is given by*

$$\mathcal{L}_{(Y,\Sigma)} = \left(r - \frac{1}{2}Tr(\Sigma_t)\right)\frac{\partial}{\partial y} + \frac{1}{2}\mathbf{e}'\Sigma_t\mathbf{e}\frac{\partial^2}{\partial y^2} + \mathcal{L}_{\Sigma}.$$

Proof: Under the independence assumption of $\mathbf{W}_t$ and Z_t the covariation $< \Sigma_{ij}, Y >= 0$ for all t and for all $i, j = 1, .., n$. The other terms are straightforward from Proposition 5.6 and from Itô's formula. Note, $d < Y_t >= \mathbf{e}'\Sigma_t\mathbf{e}dt$. □

Following [Duffie et al., 2000], in order to solve the pricing problem for plain vanilla options, we need the Laplace transform of the process in (5.10). Laplace transforms of Wishart processes are exponentially affine (See [Bru, 1991]). We guess that the moment generating function of asset returns is exponential affine in a combination of Y and the elements of the Wishart matrix. Hence, we are looking for three deterministic functions $A(t) \in M_n$, $b(t) \in \mathbb{R}$ and $c(t) \in \mathbb{R}$ that parameterize the Laplace transform:

$$\begin{aligned}\Psi_{\gamma,t}(\tau) &= \mathbb{E}_t[\exp(\gamma Y_{t+\tau})] \qquad (5.11)\\ &= \exp(\mathrm{Tr}[A(\tau)\Sigma_t] + b(\tau)Y_t + c(\tau)),\end{aligned}$$

where $\gamma \in \mathbb{R}$ and $\mathbb{E}_t[\cdot]$ is the conditional risk-neutral expectation. By applying the Feynman-Kač argument, we have

$$\frac{\partial \Psi_{\gamma,t}}{\partial \tau} = \mathcal{L}_{Y,\Sigma}\Psi_{\gamma,t},$$
$$\Psi_{\gamma,t}(0) = \exp(\gamma Y_t),$$

that is

$$\frac{\partial \Psi_{\gamma,t}}{\partial \tau} = \left(r - \frac{1}{2}\mathrm{Tr}(\Sigma_t)\right)\frac{\partial \Psi_{\gamma,t}}{\partial y} + \frac{1}{2}\mathbf{e}'\Sigma_t\mathbf{e}\frac{\partial^2 \Psi_{\gamma,t}}{\partial y^2}$$
$$+ \mathrm{Tr}\left[(\Omega\Omega' + M\Sigma + \Sigma M')D\Psi_{\gamma,t} + 2(\Sigma D Q' Q D)\Psi_{\gamma,t}\right].$$

By replacing the candidate in (5.11), we obtain

$$\mathrm{Tr}[\frac{\partial}{\partial \tau}A(\tau)\Sigma] + \frac{\partial}{\partial \tau}b(\tau)Y_t + \frac{\partial}{\partial \tau}c(\tau) = \left(r - \frac{1}{2}\mathrm{Tr}(\Sigma_t)\right)b(\tau) + \frac{1}{2}\mathbf{e}'\Sigma_t\mathbf{e}b(\tau)^2$$
$$+ \mathrm{Tr}\left[(\Omega\Omega' + M\Sigma + \Sigma M')A(\tau) + 2\Sigma A(\tau)Q'QA(\tau)\right],$$

with boundary conditions

$$A(0) = 0 \in M_n,$$
$$b(0) = \gamma \in \mathbb{R},$$
$$c(0) = 0 \in \mathbb{R}.$$

By identifying the coefficients of Y_t and integrating the ODE, we obtain

$$b(\tau) = \gamma \qquad \forall \tau.$$

We should be aware that we can use the identity $f = \mathrm{Tr}[\frac{f}{n}\mathbf{ee}']$ for a real-valued function f. Thus, we identify the coefficients of Σ by

$$\begin{aligned}
\mathrm{Tr}\left[\frac{\partial}{\partial\tau}A(\tau)\Sigma_t\right] &= -\frac{\gamma}{2}\mathrm{Tr}[\Sigma_t] + \frac{\gamma^2}{2n}\mathrm{Tr}[\mathbf{e}'\Sigma_t\mathbf{eee}'] + \mathrm{Tr}\left[(M\Sigma_t + \Sigma_t M')A(\tau) + 2\Sigma_t A(\tau)Q'QA(\tau)\right] \\
&= \mathrm{Tr}\left[-\frac{\gamma}{2}\Sigma_t + \frac{\gamma^2}{2n}\mathbf{ee}'\Sigma_t\mathbf{ee}' + (M\Sigma_t + \Sigma_t M')A(\tau) + 2\Sigma_t A(\tau)Q'QA(\tau)\right] \\
&= \mathrm{Tr}\left[-\frac{\gamma}{2}\Sigma_t + \frac{\gamma^2}{2n}\mathbf{ee}'\mathbf{ee}'\Sigma_t + A(\tau)M\Sigma_t + M'A(\tau)\Sigma_t + 2A(\tau)Q'QA(\tau)\Sigma_t\right] \\
&= \mathrm{Tr}\left[-\frac{\gamma}{2}\Sigma_t + \frac{\gamma^2}{2}\mathbf{ee}'\Sigma_t + A(\tau)M\Sigma_t + M'A(\tau)\Sigma_t + 2A(\tau)Q'QA(\tau)\Sigma_t\right].
\end{aligned}$$

We obtain the matrix Riccati ODE by

$$\begin{aligned}
\frac{\partial}{\partial\tau}A(\tau) &= \frac{\gamma^2}{2}\mathbf{ee}' - \frac{\gamma}{2}\mathbf{1}_n + A(\tau)M + M'A(\tau) + 2A(\tau)Q'QA(\tau), \\
A(0) &= 0.
\end{aligned}$$

The solution can be derived analogue to the approach in [Fonseca et al., 2005]. We obtain

$$A(\tau) = A_{22}(\tau)^{-1}A_{21}(\tau),$$

where

$$\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) \end{pmatrix} = \exp\left\{\tau\begin{pmatrix} M & -2Q'Q \\ \frac{\gamma^2}{2}\mathbf{ee}' - \frac{\gamma}{2}\mathbf{1}_n & -M' \end{pmatrix}\right\}.$$

Finally, the function $c(\tau)$ follows by direct integration. We get

$$\begin{aligned}
\frac{\partial}{\partial\tau}c(\tau) &= \mathrm{Tr}[\Omega\Omega' A(\tau)] + \gamma r, \\
c(0) &= 0.
\end{aligned}$$

We need the characteristic function, which can be derived by replacing γ by $i\gamma$ ($i = \sqrt{-1}$) in the moment generating function, in order to price options with stochastic Wishart covariance.

Proposition 5.7 *The characteristic function of the asset returns is given by*

$$\begin{aligned}\Phi_{\gamma,t}(\tau) &= \Psi_{i\gamma,t}(\tau)\\ &= \exp\left(\operatorname{Tr}\left[A(\tau)\Sigma_t\right] + i\gamma Y_t + c(\tau)\right),\end{aligned}$$

where the deterministic functions $A(\tau) \in M_n(\mathbb{C})$ *and* $c(\tau) \in \mathbb{C}$ *are given by*

$$\begin{aligned}A(\tau) &= A_{22}(\tau)^{-1}A_{21}(\tau),\\ c(\tau) &= \operatorname{Tr}\left[\Omega\Omega'\int_0^\tau A(s)ds\right] + i\gamma r\tau,\end{aligned}$$

with

$$\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) \end{pmatrix} = \exp\left\{\tau\begin{pmatrix} M & -2Q'Q \\ \frac{-\gamma^2}{2}\mathbf{ee}' - \frac{i\gamma}{2}\mathbf{1}_n & -M' \end{pmatrix}\right\}.$$

Proof: We replace γ by $i\gamma$ and calculate straightforward. □

It is known (see e.g. [Scott, 1997]) that the initial option value is determined as

$$C = S\Pi_1 - Ke^{-rT}\Pi_2,$$

where we assume that the interest rate r is constant. Π_1 and Π_2 are given by:

$$\begin{aligned}\Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left(\frac{e^{-iu\log K}\Phi_{u-i,0}(T)}{iu\Phi_{-i,0}(T)}\right)du,\\ \Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left(\frac{e^{-iu\log K}\Phi_{u,0}(T)}{iu}\right)du.\end{aligned}$$

$\operatorname{Re}(\cdot)$ denotes the real part of a complex number.

Remark 5.8 *We can not proceed in the same way for arithmetic basket options as for the geometric basket options since the dynamic of* $Y_t = \log B_t$ *is not affine in* W_t*. Therefore, to assume the Laplace transform to be exponentially affine is bad.*

5.4 Calibration data to the Wishart Process

The process given in (5.9) is difficult to calibrate since there are too many degrees of freedom to determine. Furthermore, we face the problem of loosing positive definiteness in discrete simulation for large time-steps. In order to solve these problems, we analyze a general autoregressive Wishart (WAR) process.

Let $\mathbf{x}_t$ denote a vector-autoregressive (VAR) process of order one and dimension n. This process satisfies

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + \boldsymbol{\epsilon}_{t+1}, \tag{5.12}$$

where $(\boldsymbol{\epsilon}_t)$ is a sequence of iid random vectors with multivariate Gaussian distribution $N(\mathbf{0}, \Lambda)$ and Λ is assumed to be positive definite. $(\mathbf{x}_t)$ is stationary if the matrix A has eigenvalues with modulus less than one. (Here A is different from A in Section 5.3). The WAR process Σ_t of order one is defined by

$$\Sigma_t = \sum_{k=1}^{K} \mathbf{x}_{k,t}\mathbf{x}'_{k,t},$$

where the processes $\mathbf{x}_{k,t}$, $k = 1, \ldots, K$, are independent Gaussian VAR(1), given in (5.12). The next proposition gives us first and second order conditional moments of the WAR(1) process.

Proposition 5.9 *We have*

$$\mathbb{E}_t[\Sigma_{t+1}] = A\Sigma_t A' + K\Lambda.$$

Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ be four vectors in $\mathbb{R}^n$. We get

$$\begin{aligned} cov_t(\boldsymbol{\gamma}'\Sigma_{t+1}\boldsymbol{\alpha}, \boldsymbol{\delta}'\Sigma_{t+1}\boldsymbol{\beta}) = {} & \boldsymbol{\gamma}'A\Sigma_t A'\boldsymbol{\delta}\boldsymbol{\alpha}'\Lambda\boldsymbol{\beta} + \boldsymbol{\gamma}'A\Sigma_t A'\boldsymbol{\beta}\boldsymbol{\alpha}'\Lambda\boldsymbol{\delta} + \boldsymbol{\alpha}'A\Sigma_t A'\boldsymbol{\delta}\boldsymbol{\gamma}'\Lambda\boldsymbol{\beta} \\ & + \boldsymbol{\alpha}'A\Sigma_t A'\boldsymbol{\beta}\boldsymbol{\gamma}'\Lambda\boldsymbol{\delta} + K[\boldsymbol{\gamma}'\Lambda\boldsymbol{\beta}\boldsymbol{\alpha}'\Lambda\boldsymbol{\delta} + \boldsymbol{\alpha}'\Lambda\boldsymbol{\beta}\boldsymbol{\gamma}'\Lambda\boldsymbol{\delta}]. \end{aligned}$$

Proof: See [Gourieroux et al., 2004]. □

The first and second order conditional moments are affine functions of the lagged values of the volatility process. In particular, the WAR(1) process is a weak linear AR(1) process. We have

$$\Sigma_{t+1} = A\Sigma_t A' + K\Lambda + \eta_{t+1}, \tag{5.13}$$

where η_{t+1} is a matrix of stochastic errors with conditional mean zero. But what is the relationship between the dynamic in (5.9) and the WAR(1) process?

From [Gourieroux et al., 2004] we know that if A can be written as $A = \exp(M)$, where M is a matrix, the WAR(1) process is a time-discretized diffusion process and can be written in the form of (5.9). Moreover, if K is an integer the diffusion process is obtained by summing the squares of K independent multivariate Ornstein-Uhlenbeck processes.

The calibration of the WAR(1) process is much simpler than the calibration of the process in (5.9). Let $\Sigma_1, \ldots, \Sigma_T$ denote a time-series of volatility matrices.

Proposition 5.10 *Let us assume $K \geq n$.*

a) K and Λ are identifiable whereas the autoregressive coefficient A is identifiable up to its sign.

b) Λ is first-order identifiable up to a scale factor and A is first-order identifiable up to its sign. The degree of freedom K is second order identifiable.

Proof: See [Gourieroux et al., 2004]. □

We mean by first and second order identifiable that something is identifiable from the first respectively second order conditional moments. Let us define $\Lambda^* := K\Lambda$. The first-order conditional moments can be used to calibrate the parameters A and Λ up to sign and scale factor. The first order method of moments is equivalent to nonlinear least squares. The ordinary nonlinear least squares estimators are defined as

$$\left(\hat{A}, \hat{\Lambda}^*\right) = \underset{A,\Lambda^*}{\operatorname{argmin}}\, S^2(A, \Lambda^*),$$

where

$$S^2(A, \Lambda^*) = \sum_{t=2}^{T} \|\mathrm{vech}(\Sigma_t) - \mathrm{vech}(A\Sigma_{t-1}A' + \Lambda^*)\|^2,$$

and $\mathrm{vech}(\Sigma_t)$ denotes the vector obtained by stacking the lower triangular elements of Σ_t. To estimate the degree of freedom K, from where we can derive Λ, we have to assume that the WAR(1) process is strictly stationary. As already mentioned this follows if A has eigenvalues with modulus less than one. Denote by $\Lambda(\infty)$ the solution of

$$\Lambda(\infty) = A\Lambda(\infty)A' + \Lambda.$$

The estimate of K follows from the following steps:

1. Compute $\hat{\Lambda}^*(\infty)$ as a solution of

$$\hat{\Lambda}^*(\infty) = \hat{A}\hat{\Lambda}^*(\infty)\hat{A}' + \hat{\Lambda}^*.$$

2. Choose a portfolio allocation $\mathbf{a}$ and compute its sample volatility

$$\hat{V}(\mathbf{a}'\Sigma_t\mathbf{a}) = \frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{a}'\Sigma_t\mathbf{a} - \frac{1}{T}\sum_{t=1} T\mathbf{a}'\Sigma_t\mathbf{a}\right]^2.$$

3. A consistent estimator of K is

$$\hat{K}(\mathbf{a}) = 2\frac{\left(\mathbf{a}'\hat{\Lambda}^*(\infty)\mathbf{a}\right)^2}{\hat{V}(\mathbf{a}'\Sigma_t\mathbf{a})}.$$

4. A consistent estimator of Λ is $\hat{\Lambda}(\mathbf{a}) = \hat{\Lambda}^*/\hat{K}(\mathbf{a})$.

Hence, we can calibrate market data to a WAR(1) process, from where we can simulate future volatility pathes and future multivariate asset prices. Compared to the dynamic in (5.12) we do not face the problem of loosing positive definiteness in discrete time-stepping.

Remark 5.11 *The WAR(1) process does not conserve market completeness. For example*

$\mathbf{a}$	$(0,1,1,1,1)'$	$(1,0,1,1,1)'$	$(1,1,0,1,1)'$	$(1,1,1,0,1)'$	$(1,1,1,1,0)'$	$(1,1,1,1,1)'$
$\hat{K}$	2.3961	3.0698	3.2714	3.4447	2.7687	2.9242

Table 4: Estimates of $\hat{K}$ for different vectors $\mathbf{a}$ according to the second order identification in Section 5.4.

$\hat{A}$					$\hat{\Lambda}$				
0.3728	0.1078	0.1145	0.2669	0.2009	0.0020	0.0006	0.0002	-0.0001	0.0015
-0.2291	0.5106	0.0604	-0.4349	0.4221	0.0006	0.0008	-0.0002	-0.0000	0.0004
-0.1700	0.3759	0.3756	-0.1683	0.3497	0.0002	-0.0002	0.0021	-0.0002	0.0000
-0.0841	0.0325	0.0342	0.2436	0.1459	-0.0001	-0.0000	-0.0002	0.0003	-0.0001
-0.1179	0.2034	0.1109	0.2272	0.6527	0.0015	0.0004	0.0000	-0.0001	0.0015

Table 5: $\hat{A}$ and $\hat{\Lambda}$ calculated by ordinary nonlinear least square. Additionally $\hat{\Lambda}$ is calibrated by $\hat{K} = 2.9242$.

we have in the case of a currency triangle in a complete market (e.g. USD, EUR, JPY[3]*) that the sum of the arcus cosines of the correlations is π. This property is lost in simulating a Wishart process almost sure.*

5.4.1 Calibrating Data to a five dimensional Basket of Currency Rates

In the following example we estimate the driving parameters in (5.13) for a five dimensional basket of currencies versus EUR. The basket contains the currencies {USD, GBP, PLN, CHF, SGD}[4]. We have daily data from January 4^{th}, 1999 until April 27^{th}, 2006. That are 1876 business days in total. The series of covariances is calculated historically on a monthly basis of 21 business days without overlapping. Hence, we have a time-series of covariance matrices of length $\lfloor 1876/21 \rfloor = 89$. The results of the estimation according to the procedure in Section 5.4 are in Table 4 and 5. The estimated degrees of freedom $\hat{K}$ are close to three. The mean is 2.9792. Therefore, one can assume that the monthly covariances are driven by a WAR(1) process with three degrees of freedom.

[3]JPY: Japanese Yen

[4]USD: U.S. Dollar, GBP: British Pound, PLN: Polish Zloty, CHF: Swiss Franc, SGD: Singapore Dollar

5.4.2 Influence of Stochastic Covariance on the Price of an Arithmetic Basket Call Option

In the next example we see the influence of a stochastic covariance matrix on the price of a basket option. We simulate an arithmetic European basket call option with stochastic covariance structure which follows a Wishart process with the calibrated parameters of Table 5 and $K = 3$. Unfortunately, the calibrated values of Λ are small such that the covariance matrix is almost constant and has no stochastic character anymore. This would lead to results which are similar to calculations without stochastic covariance structure. Therefore, we multiply Λ in the following simulation by a factor of 10. For the constant covariance structure we use the expected value of Σ_T, which can be calculated iteratively by the results of Proposition 5.9. Let us assume the time interval from 0 to T is split into $[0, \Delta T, \ldots, (N-1)\Delta T, T]$ subintervals. Then,

$$\begin{aligned}
\mathbb{E}[\Sigma_T] &= \mathbb{E}[\mathbb{E}_{(N-1)\Delta T}[\Sigma_T]] \\
&= \mathbb{E}[A\Sigma_{(N-1)\Delta T}A' + K\Lambda] \\
&= A\mathbb{E}[\Sigma_{(N-1)\Delta T}]A' + K\Lambda \\
&= A\mathbb{E}[\mathbb{E}_{(N-2)\Delta T}[\Sigma_{(N-1)\Delta T}]]A' + K\Lambda \\
&= A\mathbb{E}[A\Sigma_{(N-2)\Delta T}A' + K\Lambda]A' + K\Lambda \\
&= A^2\mathbb{E}[\Sigma_{(N-2)\Delta T}]A'^2 + AK\Lambda A' + K\Lambda \\
&= \ldots \\
&= A^N\Sigma_0 A'^N + \sum_{i=1}^{N} A^{N-1}K\Lambda A'^{N-1},
\end{aligned}$$

where $\mathbb{E}_t[\cdot]$ denotes the expectation conditional on the filtration up to time t. In the calculated example we assume that $\Sigma_0 = 0$, $T = 1$, and $N = 12$. Therefore, we obtain

$$\mathbb{E}[\Sigma_T] = \begin{pmatrix} 0.0360 & 0.0154 & 0.0141 & 0.0014 & 0.0300 \\ 0.0154 & 0.0172 & 0.0079 & 0.0014 & 0.0143 \\ 0.0141 & 0.0079 & 0.0349 & 0.0000 & 0.0127 \\ 0.0014 & 0.0014 & 0.0000 & 0.0036 & 0.0017 \\ 0.0300 & 0.0143 & 0.0127 & 0.0017 & 0.0307 \end{pmatrix},$$

for the parameters given in Table 5, $K = 3$ and Λ multiplied by a factor of 10. In Figure 10 we compare the result of a MC simulation for the arithmetic basket call with Wishart covariance matrix to the inverse gamma approximation of Proposition 3.4, where we use $\mathbb{E}[\Sigma_T]$ for the constant covariance matrix. One remarks that the option price with stochastic covariance is higher than the price with constant covariance. The largest difference occurs in the ATM region. In the OTM and ITM (in-the-money) region the difference tends to zero.

5.4.3 Introducing Correlation between the Underlying Assets and the Wishart Process

In (5.8) and (5.9) we assume independence between the Brownian Motion increments dZ_t and $\mathbf{dW}_t$. Analysis of market data shows that there can be a correlation between the underlying assets process and the covariance process. But how can we solve the challenge of introducing correlation R between a vector and a matrix Brownian motion? One solution can be derived from the VAR(1) regression equation in (5.12). The continuous time analogue of this regression is:

$$d\mathbf{x}_t = M\mathbf{x}_t + \Omega \left(R d\mathbf{W}_t + \sqrt{\mathbf{1}_n - RR'} d\mathbf{W}_t^{\perp} \right), \tag{5.14}$$

where M denotes a matrix such that $A = \exp(M)$ and Ω is such that $\Omega\Omega' = \Lambda$ in (5.9). $d\mathbf{W}_t$ is the Brownian motion of the underlying assets in (5.8) and $d\mathbf{W}_t^{\perp}$ is an independent of $d\mathbf{W}_t$ Brownian motion. Hence, we introduced a correlation structure R between the covariance

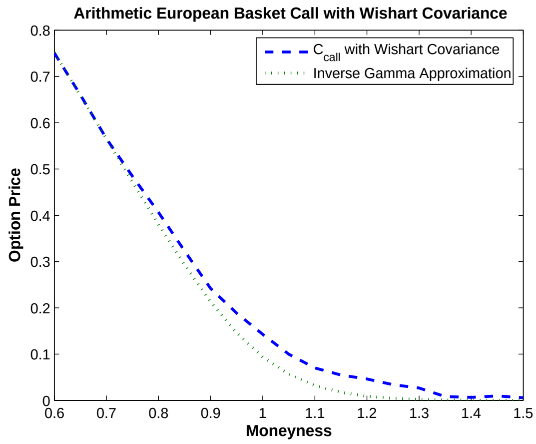

Figure 10: The price of an arithmetic European call option with stochastic Wishart covariance matrix and the price of the same option with constant covariance matrix calculated by the inverse gamma approximation of Proposition 3.4. The parameters of the Wishart process are chosen from Table 5 and $K = 3$. We multiply Λ by a factor of 10, otherwise we would not see a difference in price through the small entries of Λ. The remaining parameters are $\mathbf{S}(0) = (1.2414, 0.6958, 3.8737, 1.5814, 1.9702)'$, $\mathbf{r} = (0.0143, 0.0156, 0.0072, -0.0175, -0.0036)'$, $\mathbf{w} = (0.2, 0.2, 0.2, 0.2, 0.2)'$ and $T = 1$. For the inverse gamma approximation we choose $\mathbb{E}[\Sigma_T]$ for the constant covariance matrix. In the Wishart simulation we update the covariance matrix monthly whereas the price process is updated daily. The moneyness is defined by $K/B(0)$.

process Σ_t and the underlying assets process S_t.

6 Conclusions

In this paper, we have introduced closed-form solutions to price and hedge geometric as well as arithmetic basket options in European style. Furthermore, we have given a numerical method, based on finite differences, to price American style arithmetic basket options. This approach has been extended to derive solutions of knock-out options. Unfortunately, we have not been able to give solutions for more than two dimensional problems. The number of calculations increases enormously in the dimension. The given algorithm is in general form such that it can be used for higher dimensions.

In a second part, we have analyzed stochastic covariance matrices. First we have looked at possible dynamics for the correlation process. But for more than two underlying assets we face the problem of loosing positive semi-definiteness for the covariance matrix. Therefore, we have introduced Wishart processes to overcome this point. We have solved the geometric basket option problem with stochastic covariance by adapting an approach given in [Fonseca et al., 2005]. They assume that the logarithm of the basket is exponentially affine in the moment generating function. Finally, we have given a way to calibrate Wishart processes to market data. This approach is based on a vector-autoregressive process. In a Monte Carlo simulation we have seen that the price of an option with stochastic covariance matrix can differ from the price with constant covariance matrix. The largest difference has occurred in the at-the-money region.

Further open issues are the implementation of the finite differences approach to price American style option for more than two dimensions, static replication of basket options, and closed form solutions of arithmetic basket options with Wishart covariance matrix inclusive the purpose of hedging these vehicles.

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A Appendix

A.1 Degrees of Freedom of a Student t Fit

Another way of testing the hypothesis of normality is fitting the time-series to a Student t distribution. It is well known that the degrees of freedom $\nu \in \mathbb{N}$ of a Student t distribution are an indicator for the existing moments. In fact $\nu - 1$ is the largest existing moment. We have a normal distribution for $\nu \to \infty$. Hence, we can fit daily, weekly and monthly return series to a Student t distribution. The higher the degrees of freedom, the higher the probability of normally distributed returns.

Remark A.1 *To calculate the MLE of a Student t distribution, we have to solve a system of nonlinear equations. Let us assume we have a time-series* $\mathbf{x}$ *of random variable with distribution* $t(\nu, \mu\sigma)$*, where* μ *is the location and* σ *the scale. The log likelihood is given by*

$$\begin{aligned} l(\nu, \mu, \sigma; \mathbf{x}) = & -n\log(\sigma) + n\log\left(\Gamma\left(\frac{\nu+1}{2}\right)\right) + \frac{n\nu}{2}\log(\nu) - \frac{n}{2}\log(\pi) \\ & - n\log\left(\Gamma\left(\frac{\nu}{2}\right)\right) - \frac{\nu+1}{2}\sum_{i=1}^{n}\log(y_i), \end{aligned}$$

where n *denotes the length of* $\mathbf{x}$*,* $y_i = \nu + z_i^2$ *and* $z_i = (x_i - \mu)/\sigma$*. To compute* $\boldsymbol{\Theta}_{ML} = (\hat{\nu}_{ML}, \hat{\mu}_{ML}, \hat{\sigma}_{ML})$ *we have to solve*

$$\left(\frac{\partial l(\nu, \mu, \sigma; \mathbf{x})}{\partial \mu}, \frac{\partial l(\nu, \mu, \sigma; \mathbf{x})}{\partial \sigma}, \frac{\partial l(\nu, \mu, \sigma; \mathbf{x})}{\partial \nu}\right) = \mathbf{0},$$

where

$$\begin{aligned} \frac{\partial l(\nu, \mu, \sigma; \mathbf{x})}{\partial \mu} &= \frac{\nu+1}{\sigma}\sum_{i=1}^{n}\frac{z_i}{y_i}, \\ \frac{\partial l(\nu, \mu, \sigma; \mathbf{x})}{\partial \sigma} &= -\frac{n}{\sigma} + \frac{\nu+1}{\sigma}\sum_{i=1}^{n}\frac{z_i^2}{y_i}, \\ \frac{\partial l(\nu, \mu, \sigma; \mathbf{x})}{\partial \nu} &= \frac{n}{2}\Psi\left(\frac{\nu+1}{2}\right) + \frac{n}{2}(1+\log(\nu)) - \frac{n}{2}\Psi\left(\frac{\nu}{2}\right) - \frac{\nu+1}{2}\sum_{i=1}^{n}y_i^{-1} - \frac{1}{2}\sum_{i=1}^{n}\log(y_i). \end{aligned}$$

The function $\Psi(.)$ *is called the digamma function and is the first derivative of the log of the*

Ccy	ν_{daily}	ν_{weekly}	$\nu_{monthly}$	Ccy	ν_{daily}	ν_{weekly}	$\nu_{monthly}$
USD	9.5732	79.2561	99.9983	CHF	2.6769	9.3304	32.3297
JPY	4.6550	6.7745	10.2182	ISK	3.6183	4.3584	3.8840
CZK	2.3569	5.1188	3.5302	NOK	2.3499	6.0001	4.1215
GBP	6.5180	16.1120	61.9006	AUD	5.0618	7.1229	6.6087
HUF	2.3395	2.0673	2.1344	CAD	6.7994	16.5822	99.9990
LTL	1.0022	1.0000	1.0044	HKD	9.7020	64.5730	99.9998
PLN	4.8997	5.2395	12.5633	KRW	6.5173	5.0861	5.5381
SEK	2.3538	3.7897	4.6007	NZD	5.6223	8.9963	8.8224
SIT	3.7654	7.9987	3.5861	SGD	5.9833	14.3350	99.9992
SKK	2.4624	3.1949	6.0631	ZAR	3.8274	6.0872	8.1693

Table 6: Degrees of freedom ν for different currencies versus EUR. Calculations are based on daily data from January 4^{th}, 1999 until April 27^{th}, 2006.

gamma function.

In Table 6 we find numerical calculations, based on MLE, for ν. Note, ν lies between 0 and 100 because of the numerical implementation. Generally, the longer the tenor, the more moments exist. This confirms the results in Section A.2. Another interesting observation are the small ν's for European currencies. Especially EURCHF, EURSEK and EURNOK[5] shows for daily returns ν's smaller then three, which would indicates that not even the second moment exists. Probably this is a result from an adjustment of the European currencies versus EUR.

A.2 The HKKP Tail-Index Estimator

When a distribution $F(\cdot)$ fulfills the following regular variation condition at infinity

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha},$$

then the tail-index α measures the speed with whom the tail of $F(\cdot)$ approaches zero. It implies the larger α, the less fat-tailed the distribution. Furthermore, α represents the maximum number of existing moments of a random variable with distribution $F(\cdot)$. Let us

[5]EUR: Euro, CHF: Swiss Franc, SEK: Swedish Krona, NOK: Norwegian Krone

define $\gamma := 1/\alpha$. Hill proposes in [Hill, 1975] the following estimator for the tail-index:

$$\gamma(k) = \frac{1}{k}\sum_{j=1}^{k}\log(x_{n-j+1}) - \log(x_{n-k}),$$

where x_i is the i^{th} increasing order statistic $(i = 1, \ldots, n)$ and where k is the pre-specified number of tail observation to include $k = (1, \ldots, n-1)$. Assuming the random variables belong to a class of distribution functions of the following form

$$F(x) = 1 - ax^{-\alpha}(1 + bx^{-\beta}),$$

where $\alpha, \beta > 0$ and $a, b \in \mathbb{R}$, it has been shown that the Hill estimator can be approximated by

$$\mathbb{E}[\gamma(k)] \approx \frac{1}{\alpha} - \frac{b\beta}{\alpha(\alpha+\beta)}a^{-\beta/\alpha}\left(\frac{k}{n}\right)^{\beta/\alpha},$$

and

$$\text{var}(\gamma(k)) \approx \frac{1}{k\alpha^2}.$$

Hence, the bias increases with k, while the variance decreases. For large n the bias is small. But often we face the problem of having only a small sample. [Huisman et al., 2001] states that $\gamma(k)$ is almost linearly in k and can be approximated by

$$\gamma(k) = \beta_0 + \beta_1 k + \epsilon(k), \tag{A.1}$$

where $k \leq \kappa$ and κ is a threshold value of n (e.g. $k < n/2$). Note, the error term $\epsilon(k)$ is heteroscedastic. By dividing the left- and right hand sides of (A.1) by $\sqrt{k}$ we pay regard to the heteroscedasticity and can apply Ordinary Least Squares to determine β_0 and β_1. The estimated tail-index γ equals β_0.

In Table 7 and 8 we calculate γ for different currencies versus EUR and for different tenors {weekly, daily, monthly}. The main observation is the longer the tenor the more possible are normally distributed returns. In general, γ decreases from daily to monthly data. Negative values are possible since we obtained the values from (A.1). This number is meaning less. It

Ccy	γ_{daily}	κ	n	γ_{weekly}	κ	n	γ_{mothly}	κ	n
Positive returns									
USD	0.1963	459	1875	0.1466	92	375	-0.0031	18	89
JPY	0.2244	479	1875	0.0930	99	375	0.2637	23	89
CYP	0.4307	332	1875	0.1420	81	375	0.0529	20	89
CZK	0.2105	441	1875	0.0814	81	375	0.1549	17	89
DKK	0.1475	451	1875	0.0743	95	375	-0.0069	21	89
GBP	0.1794	446	1875	0.1152	89	375	0.0576	19	89
HUF	0.2888	462	1875	0.1771	89	375	0.0467	24	89
LTL	0.1875	292	1875	0.0825	59	375	-0.3373	10	89
LVL	0.2253	433	1875	0.1180	90	375	0.0511	19	89
PLN	0.1734	443	1875	0.2151	82	375	0.1394	20	89
SEK	0.1558	461	1875	0.0738	91	375	0.2098	23	89
SIT	0.2989	513	1875	0.1194	127	375	0.0956	32	89
SKK	0.1532	414	1875	0.1370	82	375	0.0269	16	89
CHF	0.2086	470	1875	0.0992	88	375	0.0448	20	89
ISK	0.2218	456	1875	0.2787	90	375	0.2610	21	89
NOK	0.1724	441	1875	0.1189	80	375	0.0975	16	89
AUD	0.1464	442	1875	0.1688	92	375	0.0615	19	89
CAD	0.1941	441	1875	0.2143	84	375	0.0425	17	89
HKD	0.2004	467	1875	0.1504	92	375	0.0119	19	89
KRW	0.1971	450	1875	0.1336	87	375	0.1828	20	89
NZD	0.1210	434	1875	0.1732	90	375	0.2025	20	89
SGD	0.2200	453	1875	0.0935	91	375	0.0689	22	89
ZAR	0.2097	441	1875	0.1200	82	375	0.1319	18	89

Table 7: HKKP tail index estimator γ for positive returns of different currencies versus EUR. Calculations are based on daily data from January 4^{th}, 1999 until April 27^{th}, 2006.

only indicates that the Hill estimator $\gamma(k)$ has a large increase with k. Considering γ_{daily}, often only the first four moments exists. For EURCYP[6] we not even have a third moment for positive returns.

A.3 Autocorrelation in Return Series for Different Currencies

Beside the heavy-tailedness of historical FX return data, we face the assumption of iid returns. Note, independence implies zero autocorrelation but zero auto-correlation does not imply independence. For that reason we calculate the autocorrelation of historical returns as well as the auto-correlation of absolute historical returns. For independent (absolute) returns the

[6] CYP: Cyprus Pound

Ccy	γ_{daily}	κ	n	γ_{weekly}	κ	n	γ_{mothly}	κ	n
Negative returns									
USD	0.0863	467	1875	0.0736	94	375	0.1778	24	89
JPY	0.1860	449	1875	0.1012	87	375	0.0590	20	89
CYP	0.4194	311	1875	0.1866	82	375	0.1215	21	89
CZK	0.1746	486	1875	0.0805	105	375	0.1231	26	89
DKK	0.1890	432	1875	0.0682	86	375	0.2316	22	89
GBP	0.1377	475	1875	0.1099	97	375	0.0253	24	89
HUF	0.1478	459	1875	0.1275	96	375	0.0729	19	89
LTL	0.0676	334	1875	0.0776	75	375	0.1452	20	89
LVL	0.1389	429	1875	0.0676	85	375	0.2114	22	89
PLN	0.1622	489	1875	0.0832	104	375	0.0272	23	89
SEK	0.1621	471	1875	0.1853	95	375	0.4485	20	89
SIT	0.3131	357	1875	0.0810	54	375	0.0327	10	89
SKK	0.2035	507	1875	0.2254	101	375	0.0103	27	89
CHF	0.2456	451	1875	0.1347	96	375	0.1398	23	89
ISK	0.1669	464	1875	0.1597	94	375	0.0245	22	89
NOK	0.0789	487	1875	0.2135	105	375	0.0689	27	89
AUD	0.1210	488	1875	0.1088	94	375	0.0827	24	89
CAD	0.0814	489	1875	0.1063	102	375	0.0950	26	89
HKD	0.0736	463	1875	0.0706	94	375	0.1801	24	89
KRW	0.0919	483	1875	0.1317	99	375	0.0099	23	89
NZD	0.1177	495	1875	0.1448	96	375	0.0476	23	89
SGD	0.0788	478	1875	0.0823	95	375	0.1138	21	89
ZAR	0.2660	492	1875	0.1338	104	375	0.1282	25	89

Table 8: HKKP tail index estimator γ for negative returns of different currencies versus EUR. Calculations are based on daily data from January 4^{th}, 1999 until April 27^{th}, 2006.

auto-correlation is zero for all lags $k > 1$. Figure 11 - 14 show the auto-correlation diagram for daily, weekly and monthly returns for different currency rate versus EUR. The historical data are chosen from January 4^{th}, 1999 until April 27^{th}, 2006. The horizontal lines indicate the 95% upper and lower confidence bounds. The lag for $k = 0$ is obviously one and is therefore not shown.

The results indicate that there is a strong dependence in time. Nearly all returns show absolute auto-correlation for daily data. For weekly and monthly data this effect is much weaker which strengths (but not proofs it!) the independence assumption. Interestingly, the effective returns show less auto-correlation. Hence, we have a nice example of zero-autocorrelation but non-independence. The same considerations can be done for several arithmetic baskets with weights $\mathbf{w} = (1, \ldots, 1)'$. We structured four baskets:

1. Major = {USD, JPY, GBP, CHF}[7],
2. Central Europe = {HUF, CZK, PLN}[8],
3. Asia = {HKD, KRW, SGD}[9],
4. World = {USD, JPY, HUF, AUD, HKD}[10],

where the numeraire is EUR. The auto-correlation diagrams are in Figure 15 - 18. The local baskets show strong auto-correlation for absolute daily returns. On the opposite the World basket has much less absolute auto-correlation. Probably this shows a certain degree of local diversification. In general we can gather from this analysis that the basket can, depending on his structure, in- or decrease the probability of independence.

[7] USD: U.S. Dollar, JPY: Japanese Yen, GBP: British Pound, CHF: Swiss Franc

[8] HUF: Hungarian Forint, CZK: Czech Koruna Yen, PLN: Polish Zloty

[9] HKD: Hong Kong Dollar, KRW: South Korean Won, SGD: Singapore Dollar

[10] USD: U.S. Dollar, JPY: Japanese Yen, HUF: Hungarian Forint, AUD: Australian Dollar, HKD: Hong Kong Dollar

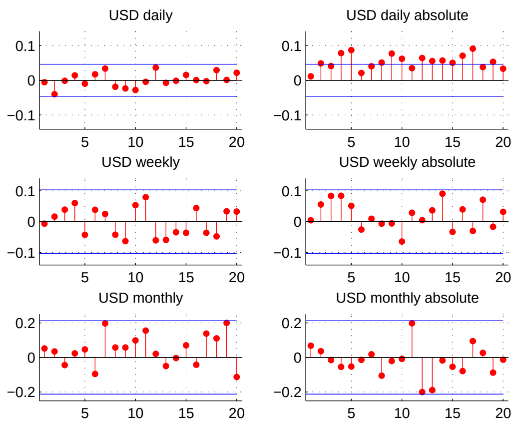

Figure 11: Historical auto-correlation diagram of the FX rate returns and absolute returns for EUR/USD from January 4^{th}, 1999 until April 27^{th}, 2006. The horizontal lines indicate the 95% upper and lower confidence bounds.

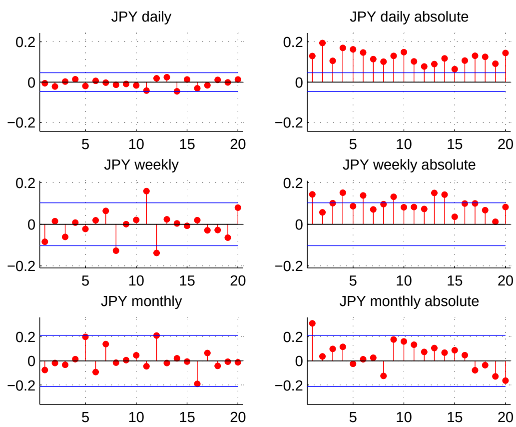

Figure 12: Historical auto-correlation diagram of the FX rate returns and absolute returns for EUR/JPY from January 4^{th}, 1999 until April 27^{th}, 2006. The horizontal lines indicate the 95% upper and lower confidence bounds.

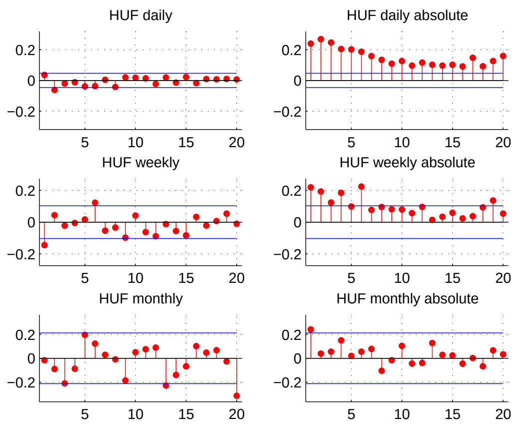

Figure 13: Historical auto-correlation diagram of the FX rate returns and absolute returns for EUR/HUF from January 4^{th}, 1999 until April 27^{th}, 2006. The horizontal lines indicate the 95% upper and lower confidence bounds.

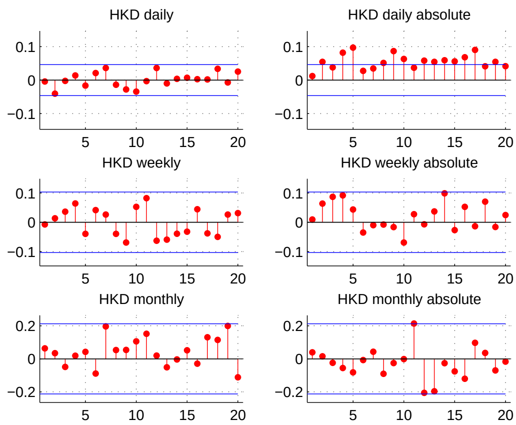

Figure 14: Historical auto-correlation diagram of the FX rate returns and absolute returns for EUR/HKD from January 4^{th}, 1999 until April 27^{th}, 2006. The horizontal lines indicate the 95% upper and lower confidence bounds.

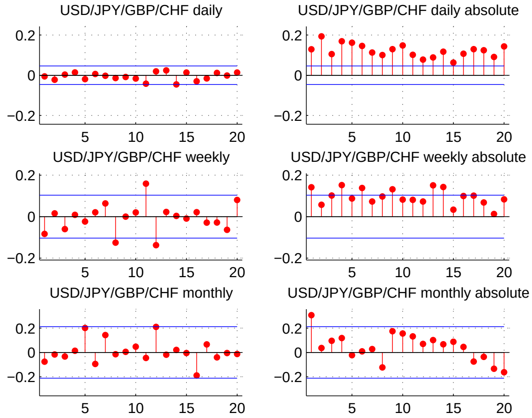

Figure 15: Historical auto-correlation diagram of the FX basket consisting of {USD, JPY, GBP, CHF} versus EUR. Data from January 4^{th}, 1999 until April 27^{th}, 2006. The horizontal lines indicate the 95% upper and lower confidence bounds.

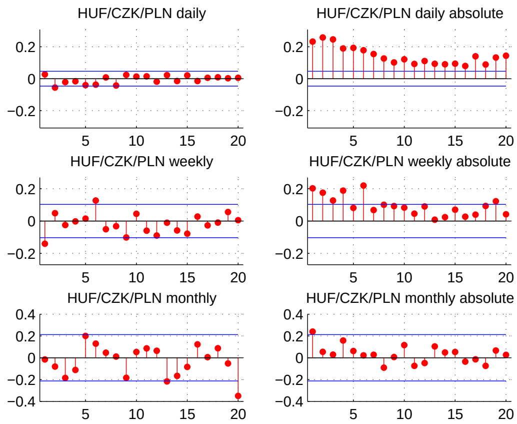

Figure 16: Historical auto-correlation diagram of the FX basket consisting of {HUF, CZK, PLN} versus EUR. Data from January 4^{th}, 1999 until April 27^{th}, 2006. The horizontal lines indicate the 95% upper and lower confidence bounds.

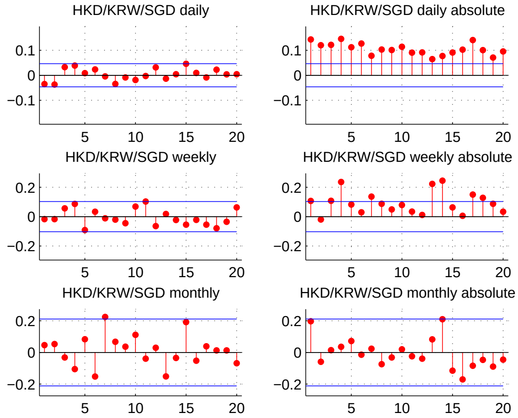

Figure 17: Historical auto-correlation diagram of the FX basket consisting of {HKD, KRW, SGD} versus EUR. Data from January 4^{th}, 1999 until April 27^{th}, 2006. The horizontal lines indicate the 95% upper and lower confidence bounds.

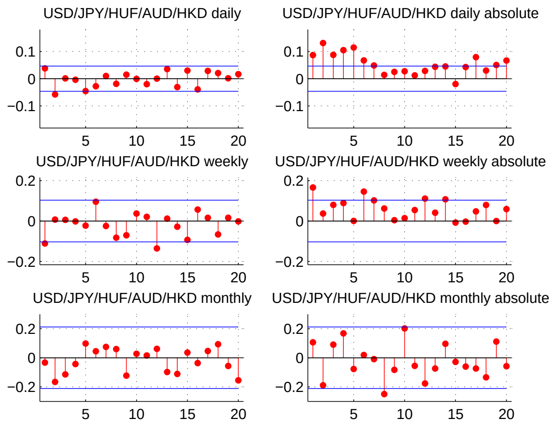

Figure 18: Historical auto-correlation diagram of the FX basket consisting of {USD, JPY, HUF, AUD, HKD} versus EUR. Data from January 4^{th}, 1999 until April 27^{th}, 2006. The horizontal lines indicate the 95% upper and lower confidence bounds.