ACCRUAL SWAPS AND RANGE NOTES

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Abstract. Here we present the standard methodology for pricing accrual swaps, range notes, and callable accrual swaps and range notes.

Key words. range notes, time swaps, accrual notes

1. Introduction.

1.1. Notation. In our notation today is always $t = 0$, and

$$D(T) = \text{today's discount factor for maturity } T. \tag{1.1a}$$

For any date t in the future, let $Z(t;T)$ be the value of \$1 to be delivered at a later date T:

$$Z(t;T) = \text{zero coupon bond, maturity } T\text{, as seen at } t. \tag{1.1b}$$

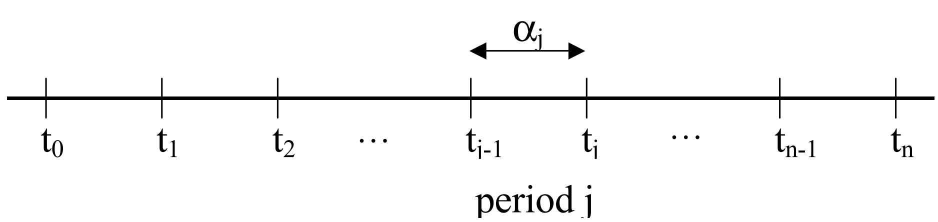
These discount factors and zero coupon bonds are the ones obtained from the currency's swap curve. Clearly $D(T) = Z(0;T)$. We use distinct notation for discount factors and zero coupon bonds to remind ourselves that discount factors $D(T)$ are *not* random; we can always obtain the current discount factors from the stripper. Zero coupon bonds $Z(t;T)$ *are* random, at least until time catches up to date t. Let

$$f_0(T) = \text{today's instantaneous forward rate for date } T, \tag{1.2a}$$
$$f(t;T) = \text{instantaneous forward rate for date } T\text{, as seen at } t. \tag{1.2b}$$

These are defined via

$$D(T) = e^{-\int_0^T f_0(T')dT'}, \qquad Z(t;T) = e^{-\int_t^T f(t,T')dT'}. \tag{1.2c}$$

1.2. Accrual swaps (fixed).



Coupon leg schedule

Fixed coupon accrual swaps (*aka* time swaps) consist of a coupon leg swapped against a funding leg. Suppose that the agreed upon reference rate is, say, k month Libor. Let

$$t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n \tag{1.3}$$

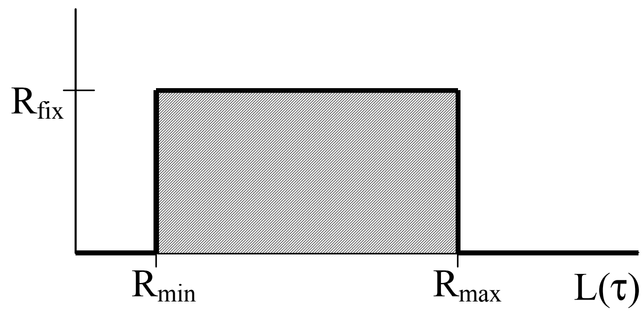


Fig. 1.1. *Daily coupon rate*

be the schedule of the coupon leg, and let the nominal fixed rate be R_{fix}. Also let $L(\tau_{st})$ represent the k month Libor rate fixed for the interval starting at τ_{st} and ending at $\tau_{end}(\tau_{st}) = \tau_{st} + k$ months. Then the coupon paid for period j is

$$C_j = \alpha_j R_{fix} \theta_j \qquad \text{paid at } t_j, \tag{1.4a}$$

where

$$\alpha_j = \text{cvg}(t_{j-1}, t_j) = \text{ day count fraction for } t_{j-1} \text{ to } t_j, \tag{1.4b}$$

and

$$\theta_j = \frac{\#\text{days } \tau_{st} \text{ in the interval with } R_{\min} \le L(\tau_{st}) \le R_{\max}}{M_j}. \tag{1.4c}$$

Here M_j is the total number of days in interval j, and $R_{\min} \le L(\tau_{st}) \le R_{\max}$ is the agreed-upon *accrual range*. Said another way, each day τ_{st} in the j^{th} period contibutes the amount

$$\frac{\alpha_j R_{fix}}{M_j} \begin{cases} 1 & \text{if } R_{\min} \le L(\tau_{st}) \le R_{\max} \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

to the coupon paid on date t_j.

For a standard deal, the leg's schedule is contructed like a standard swap schedule. The *theoretical dates* (aka *nominal dates*) are constructed monthly, quarterly, semi-annually, or annually (depending on the contract terms) backwards from the "theoretical end date." Any odd coupon is a stub (short period) at the front, unless the contract explicitly states long first, short last, or long last. The modified following business day convention is used to obtain the *actual dates* t_j from the theoretical dates. The coverage (day count fraction) is *adjusted*, that is, the day count fraction for period j is calculated from the actual dates t_{j-1} and t_j, not the theoretical dates. Also, $L(\tau_{st})$ is the fixing that pertains to periods starting on date τ_{st}, regardless of whether τ_{st} is a good business day or not. I.e., the rate $L(\tau_{st})$ set for a Friday start also pertains for the following Saturday and Sunday.

Like all fixed legs, there are many variants of these coupon legs. The only variations that do not make sense for coupon legs are "set-in-arrears" and "compounded." There are three variants that occur relatively frequently:

Floating rate accrual swaps. Minimal coupon accrual swaps. Floating rate accrual swaps are like ordinary accrual swaps except that at the start of each period, a floating rate is set, and this rate plus a margin is used in place of the fixed rate R_{fix}. Minimal coupon accrual swaps receive one rate each day Libor sets within the range and a second, usually lower rate, when Libor sets outside the range

$$\frac{\alpha_j}{M_j}\begin{cases} R_{fix} & \text{if } R_{\min} \le L(\tau_{st}) \le R_{\max} \\ R_{floor} & \text{otherwise} \end{cases} .$$

(A standard accrual swap has $R_{floor} = 0$. These deals are analyzed in Appendix B.

Range notes. In the above deals, the funding leg is a standard floating leg plus a margin. A range note is a bond which pays the coupon leg on top of the principle repayments; there is no floating leg. For these deals, the counterparty's credit-worthiness is a key concern. To determine the correct value of a range note, one needs to use an *option adjusted spread* (OAS) to reflect the extra discounting reflecting the counterparty's credit spread, bond liquidity, etc. See section 3.

Other indices. CMS and CMT accrual swaps. Accrual swaps are most commonly written using 1m, 3m, 6m, or 12m Libor for the reference rate $L(\tau_{st})$. However, some accrual swaps use swap or treasury rates, such as the 10y swap rate or the 10y treasury rate, for the reference rate $L(\tau_{st})$. These CMS or CMT accrual swaps are not analyzed here (yet). There is also no reason why the coupon cannot set on other widely published indices, such as 3m BMA rates, the FF index, or the OIN rates. These too will not be analyzed here.

2. Valuation. We value the coupon leg by replicating the payoff in terms of vanilla caps and floors.

Consider the j^{th} period of a coupon leg, and suppose the underlying indice is k-month Libor. Let $L(\tau_{st})$ be the k-month Libor rate which is fixed for the period starting on date τ_{st} and ending on $\tau_{end}(\tau_{st}) = \tau_{st} + k$ months. The Libor rate will be fixed on a date τ_{fix}, which is on or a few days before τ_{st}, depending on currency. On this date, the value of the contibution from day τ_{st} is clearly

$$V(\tau_{fix};\tau_{st}) = \text{payoff} = Z(\tau_{fix};t_j)\begin{cases} \dfrac{\alpha_j R_{fix}}{M_j} & \text{if } R_{\min} \le L(\tau_{st}) \le R_{\max} \\ 0 & \text{otherwise} \end{cases} , \tag{2.1}$$

where τ_{fix} the fixing date for τ_{st}. We value coupon j by replicating each day's contribution in terms of vanilla caplets/floorlets, and then summing over all days τ_{st} in the period.

Let $F_{dig}(t;\tau_{st},K)$ be the value at date t of a digital floorlet on the floating rate $L(\tau_{st})$ with strike K. If the Libor rate $L(\tau_{st})$ is at or below the strike K, the digital floorlet pays 1 unit of currency on the end date $\tau_{end}(\tau_{st})$ of the k-month interval. Otherwise the digital pays nothing. So on the fixing date τ_{fix} the payoff is known to be

$$F_{dig}(\tau_{fix};\tau_{st},K) = Z(\tau_{fix};\tau_{end})\begin{cases} 1 & \text{if } L(\tau_{st}) \le K \\ 0 & \text{otherwise} \end{cases} , \tag{2.2}$$

We can replicate the range note's payoff for date τ_{st} by going long and short digitals struck at $R_{\max}$ and $R_{\min}$. This yields,

$$\frac{\alpha_j R_{fix}}{M_j}\left[F_{dig}(\tau_{fix};\tau_{st},R_{\max}) - F_{dig}(\tau_{fix};\tau_{st},R_{\min})\right] \tag{2.3}$$

$$= Z(\tau_{fix};\tau_{end})\frac{\alpha_j R_{fix}}{M_j}\begin{cases} 1 & \text{if } R_{\min} \le L(\tau_{st}) \le R_{\max} \\ 0 & \text{otherwise} \end{cases} . \tag{2.4}$$

This is the same payoff as the range note, *except that the digitals pay off on $\tau_{end}(\tau_{st})$ instead of t_j.*

2.1. Hedging considerations. Before fixing the date mismatch, we note that digital floorlets are considered vanilla instruments. This is because they can be replicated to arbitrary accuracy by a bullish spread of floorlets. Let $F(t,\tau_{st},K)$ be the value on date t of a standard floorlet with strike K on the floating rate $L(\tau_{st})$. This floorlet pays $\beta\,[K-L(\tau_{st})]^{+}$ on the end date $\tau_{end}(\tau_{st})$ of the k-month interval. So on the fixing date, the payoff is known to be

$$F(\tau_{fix};\tau_{st},K)=\beta\,[K-L(\tau_{st})]^{+}\,Z(\tau_{fix};\tau_{end}). \tag{2.5a}$$

Here, β is the day count fraction of the period τ_{st} to τ_{end},

$$\beta=\mathrm{cvg}(\tau_{st},\tau_{end}). \tag{2.5b}$$

The bullish spread is constructed by going long $\dfrac{1}{\varepsilon\beta}$ floors struck at $K+\frac{1}{2}\varepsilon$ and short the same number struck at $K-\frac{1}{2}\varepsilon$. This yields the payoff

$$\begin{aligned}&\frac{1}{\varepsilon\beta}\left\{F(\tau_{fix};\tau_{st},K+\tfrac{1}{2}\varepsilon)-F(\tau_{fix};\tau_{st},K-\tfrac{1}{2}\varepsilon)\right\}\\ &\qquad=Z(\tau_{fix};\tau_{end})\begin{cases}1 & \text{if } L(\tau_{st})<K-\frac{1}{2}\varepsilon\\ \dfrac{1}{\varepsilon}\left(K+\frac{1}{2}\varepsilon-L(\tau_{st})\right) & \text{if } K-\frac{1}{2}\varepsilon<L(\tau_{st})<K+\frac{1}{2}\varepsilon\\ 0 & \text{if } K+\frac{1}{2}\varepsilon<L(\tau_{st})\end{cases},\end{aligned} \tag{2.6}$$

which goes to the digital payoff as $\varepsilon\to 0$. Clearly the value of the digital floorlet is the limit as $\varepsilon\to 0$ of

$$F_{cen}(t;\tau_{st},K,\varepsilon)=\frac{1}{\varepsilon\beta}\left\{F(t;\tau_{st},K+\tfrac{1}{2}\varepsilon)-F(t;\tau_{st},K-\tfrac{1}{2}\varepsilon)\right\}. \tag{2.7a}$$

Thus the bullish spread, and its limit, the digitial floorlet, are directly determined by the market prices of vanilla floors on $L(\tau_{st})$.

Digital floorlets may develop an unbounded Δ-risk as the fixing date is approached. To avoid this difficulty, most firms book, price, and hedge digital options as bullish floorlet spreads. I.e., they book and hedge the digital floorlet as if it were the spread in eq. 2.7a with ε set to 5bps or 10bps, depending on the aggressiveness of the firm. Alternatively, some banks choose to super-replicate or sub-replicate the digital, by booking and hedging it as

$$F_{\sup}(t;\tau_{st},K,\varepsilon)=\frac{1}{\varepsilon\beta}\left\{F(t;\tau_{st},K+\varepsilon)-F(t;\tau_{st},K)\right\} \tag{2.7b}$$

or

$$F_{\mathrm{sub}}(t;\tau_{st},K,\varepsilon)=\frac{1}{\varepsilon\beta}\left\{F(t;\tau_{st},K)-F(t;\tau_{st},K-\varepsilon)\right\} \tag{2.7c}$$

depending on which side they own. One should price accrual swaps in accordance with a desk's policy for using central- or super- and sub-replicating payoffs for other digital caplets and floorlets.

2.2. Handling the date mismatch. We re-write the coupon leg's contribution from day τ_{st} as

$$V(\tau_{fix};\tau_{st})=Z(\tau_{fix};\tau_{end})\frac{Z(\tau_{fix};t_j)}{Z(\tau_{fix};\tau_{end})}\begin{cases}\dfrac{\alpha_j R_{fix}}{M_j} & \text{if } R_{\min}\le L(\tau_{st})\le R_{\max}\\ 0 & \text{otherwise}\end{cases}. \tag{2.8}$$

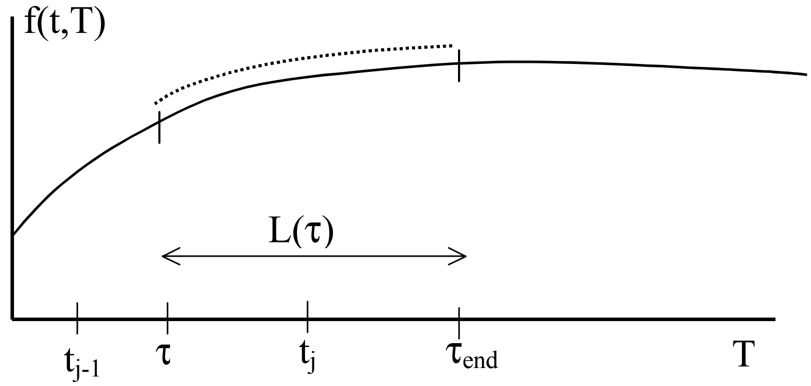


FIG. 2.1. *Date mismatch is corrected assuming only parallel shifts in the forward curve*

The ratio $Z(\tau_{fix};t_j)/Z(\tau_{fix};\tau_{end})$ is the manifestation of the date mismatch. To handle this mismatch, we *approximate* the ratio by assuming that the yield curve makes only parallel shifts over the relevent interval. See figure 2.1. So suppose we are at date t_0. Then we assume that

$$\text{(2.9a)} \qquad \frac{Z(\tau_{fix};t_j)}{Z(\tau_{fix};\tau_{end})} = \frac{Z(t_0;t_j)}{Z(t_0;\tau_{end})} e^{-[L(\tau_{st})-L_f(t_0,\tau_{st})](t_j-\tau_{end})}$$
$$= \frac{Z(t_0;t_j)}{Z(t_0;\tau_{end})} \left\{1 + [L(\tau_{st}) - L_f(t_0,\tau_{st})](\tau_{end} - t_j) + \cdots\right\}.$$

Here

$$\text{(2.9b)} \qquad L_f(t_0,\tau_{st}) \equiv \frac{Z(t_0;\tau_{st}) - Z(t_0;\tau_{end})}{\beta Z(t_0;\tau_{end})} + \mathrm{bs}(\tau_{st}),$$

is the forward rate for the k-month period starting at τ_{st}, as seen at the current date t_0, $\mathrm{bs}(\tau_{st})$ is the floating rate's basis spread, and

$$\text{(2.9c)} \qquad \beta = \mathrm{cvg}(\tau_{st}, \tau_{end}),$$

is the day count fraction for τ_{st} to τ_{end}. Since $L(\tau_{st}) = L_f(\tau_{fix}, \tau_{st})$ represents the floating rate which is actually fixed on the fixing date τ_{ex}, 2.9a just assumes that any change in the yield curve between t_j and τ_{end} is the same as the change $L_f(\tau_{fix},\tau_{st}) - L_f(t_0,\tau_{st})$ in the reference rate between the observation date t_0, and the fixing date τ_{fix}. See figure 2.1.

We actually use a slightly different approximation,

$$\text{(2.10a)} \qquad \frac{Z(\tau_{fix};t_j)}{Z(\tau_{fix};\tau_{end})} \approx \frac{Z(t_0;t_j)}{Z(t_0;\tau_{end})} \frac{1+\eta\beta L(\tau_{st})}{1+\eta\beta L_f(t_0,\tau_{st})}$$

where

$$\text{(2.10b)} \qquad \eta = \frac{\tau_{end} - t_j}{\tau_{end} - \tau_{st}}.$$

We prefer this approximation because it is the only linear approximation which accounts for the day count basis correctly, is exact for $\tau_{st} = t_j$, and is centerred around the current forward value for the range note.

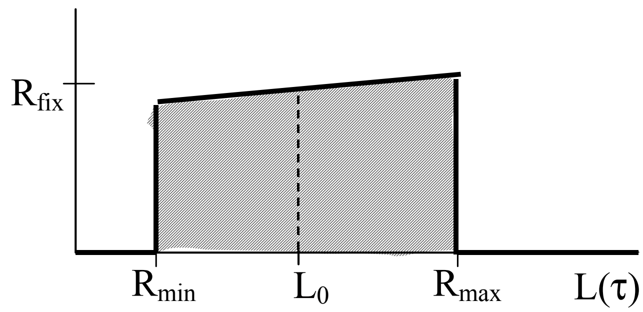


FIG. 2.2. *Effective contribution from a single day τ, after accounting for the date mis-match.*

With this approximation, the payoff from day τ_{st} is

$$V(\tau_{fix};\tau) = A(t_0,\tau_{st})Z(\tau_{fix};\tau_{end}) \begin{cases} 1+\eta\beta L(\tau_{st}) & \text{if } R_{\min} \le L(\tau_{st}) \le R_{\max} \\ 0 & \text{otherwise} \end{cases} \tag{2.11a}$$

as seen at date t_0. Here the effective notional is

$$A(t_0,\tau_{st}) = \frac{\alpha_j R_{fix}}{M_j}\frac{Z(t_0;t_j)}{Z(t_0;\tau_{end})}\frac{1}{1+\eta\beta L_f(t_0,\tau_{st})}. \tag{2.11b}$$

We can replicate this digital-linear-digital payoff by using a combination of two digital floorlets and two standard floorlets. Consider the combination

$$\begin{aligned} V(t;\tau_{st}) \equiv A(t_0,\tau_{st})\,\{(1+\eta\beta R_{\max})F_{dig}(t,\tau_{st};R_{\max}) &- (1+\eta\beta R_{\min})F_{dig}(t,\tau_{st};R_{\min}) \\ &- \eta F(t,\tau_{st};R_{\max}) + \eta F(t,\tau_{st};R_{\min}). \end{aligned} \tag{2.12}$$

Setting t to the fixing date τ_{fix} shows that this combination matches the contribution from day τ_{st} in eq. 2.11a. Therefore, this formula gives the value of the contribution of day τ_{st} for all earlier dates $t_0 \le t \le \tau_{fix}$ as well.

Alternatively, one can replicate the payoff as close as one wishes by going long and short floorlet spreads centerred around $R_{\max}$ and $R_{\min}$. Consider the portfolio

$$\begin{aligned} \tilde{V}(t;\tau_{st},\varepsilon) = \frac{A(t_0,\tau_{st})}{\varepsilon\beta}\,\{a_1(\tau_{st})F(t;\tau_{st},R_{\max}+\tfrac{1}{2}\varepsilon) &- a_2(\tau_{st})F(t;\tau_{st},R_{\max}-\tfrac{1}{2}\varepsilon) \\ - a_3(\tau_{st})F(t;\tau_{st},R_{\min}+\tfrac{1}{2}\varepsilon) &+ a_4(\tau_{st})F(t;\tau_{st},R_{\min}-\tfrac{1}{2}\varepsilon)\} \end{aligned} \tag{2.13a}$$

with

$$a_1(\tau_{st}) = 1+\eta\beta(R_{\max}-\tfrac{1}{2}\varepsilon), \qquad a_2(\tau_{st}) = 1+\eta\beta(R_{\max}+\tfrac{1}{2}\varepsilon), \tag{2.13b}$$

$$a_3(\tau_{st}) = 1+\eta\beta(R_{\min}-\tfrac{1}{2}\varepsilon), \qquad a_4(\tau_{st}) = 1+\eta\beta(R_{\min}+\tfrac{1}{2}\varepsilon). \tag{2.13c}$$

Setting t to τ_{fix} yields

$$\tilde{V} = A(t_0,\tau_{st})Z(\tau_{fix};\tau_{end}) \begin{cases} 0 & \text{if } L(\tau_{st}) < R_{\min}-\tfrac{1}{2}\varepsilon \\ 1+\eta\beta L(\tau_{st}) & \text{if } R_{\min}+\tfrac{1}{2}\varepsilon < L(\tau_{st}) < R_{\max}-\tfrac{1}{2}\varepsilon \\ 0 & \text{if } R_{\max}+\tfrac{1}{2}\varepsilon < L(\tau_{st}) \end{cases}, \tag{2.14}$$

with linear ramps between $R_{\min} - \frac{1}{2}\varepsilon < L(\tau_{st}) < R_{\min} + \frac{1}{2}\varepsilon$ and $R_{\max} - \frac{1}{2}\varepsilon < L(\tau_{st}) < R_{\max} + \frac{1}{2}\varepsilon$. As above, most banks would choose to use the floorlet spreads (with ε being 5bps or 10bps) instead of using the more troublesome digitals. For a bank insisting on using exact digital options, one can take ε to be 0.5bps to replicate the digital accurately..

We now just need to sum over all days τ_{st} in period j and all periods j in the coupon leg,

$$V_{cpn}(t) = \sum_{j=1}^{n} \sum_{\tau_{st}=t_{j-1}+1}^{t_j} \frac{A(t_0,\tau_{st})}{\varepsilon\beta} \Big\{ \big[1 + \eta\beta(R_{\max} - \tfrac{1}{2}\varepsilon)\big] F(t;\tau_{st}, R_{\max} + \tfrac{1}{2}\varepsilon) - \big[1 + \eta\beta(R_{\max} + \tfrac{1}{2}\varepsilon)\big] F(t;\tau_{st}, R_{\max} - \tfrac{1}{2}\varepsilon) - \big[1 + \eta\beta(R_{\min} - \tfrac{1}{2}\varepsilon)\big] F(t;\tau_{st}, R_{\min} + \tfrac{1}{2}\varepsilon) + \big[1 + \eta\beta(R_{\min} + \tfrac{1}{2}\varepsilon)\big] F(t;\tau_{st}, R_{\min} - \tfrac{1}{2}\varepsilon) \Big\}. \tag{2.15}$$

This formula replicates the value of the range note in terms of vanilla floorlets. These floorlet prices should be obtained directly from the marketplace using market quotes for the implied volatilities at the relevent strikes. Of course the centerred spreads could be replaced by super-replicating or sub-replicating floorlet spreads, bringing the pricing in line with the bank's policies.

Finally, we need to value the funding leg of the accrual swap. For most accrual swaps, the funding leg pays floating plus a margin. Let the funding leg dates be $\tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_{\tilde{n}}$. Then the funding leg payments are

$$\text{cvg}(\tilde{t}_{i-1}, \tilde{t}_i)[R_i^{flt} + m_i] \qquad \text{paid at } \tilde{t}_i, \qquad i = 1, 2, ..., \tilde{n}, \tag{2.16}$$

where R_i^{flt} is the floating rate's fixing for the period $\tilde{t}_{i-1} < t < \tilde{t}_i$, and the m_i is the margin. The value of the funding leg is just

$$V_{fund}(t) = \sum_{i=1}^{\tilde{n}} \text{cvg}(\tilde{t}_{i-1}, \tilde{t}_i)(r_i + m_i) Z(t; \tilde{t}_i), \tag{2.17a}$$

where, by definition, r_i is the forward value of the floating rate for period $\tilde{t}_{i-1} < t < \tilde{t}_i$:

$$r_i = \frac{Z(t;\tilde{t}_{i-1}) - Z(t;\tilde{t}_i)}{\text{cvg}(\tilde{t}_{i-1}, \tilde{t}_i) Z(t;\tilde{t}_i)} + \text{bs}'_i = r_i^{true} + \text{bs}'_i\,. \tag{2.17b}$$

Here bs'_i is the basis spread for the funding leg's floating rate, and r_i^{true} is the true (cash) rate. This sum collapses to

$$V_{fund}(t) = Z(t;t_0) - Z(t;t_{\tilde{n}}) + \sum_{i=1}^{\tilde{n}} \text{cvg}(\tilde{t}_{i-1}, \tilde{t}_i)(\text{bs}'_i + m_i) Z(t;\tilde{t}_i). \tag{2.18a}$$

If we include only the funding leg payments for $i = i_0$ to $\tilde{n}$, the value is

$$V_{fund}(t) = Z(t;\tilde{t}_{i_0-1}) - Z(t;t_{\tilde{n}}) + \sum_{i=i_0}^{\tilde{n}} \text{cvg}(\tilde{t}_{i-1}, \tilde{t}_i)(\text{bs}'_i + m_i) Z(t;\tilde{t}_i). \tag{2.18b}$$

2.2.1. Pricing notes. Caplet/floorlet prices are normally quoted in terms of Black vols. Suppose that on date t, a floorlet with fixing date t_{fix}, start date τ_{st}, end date τ_{end}, and strike K has an implied vol of $\sigma_{imp}(K) \equiv \sigma_{imp}(\tau_{st}, K)$. Then its market price is

$$F(t, \tau_{st}, K) = \beta Z(t;\tau_{end}) \left\{ K\mathcal{N}(d_1) - L(t,\tau)\mathcal{N}(d_2) \right\}, \tag{2.19a}$$

where

(2.19b) $$d_{1,2} = \frac{\log K/L(t,\tau_{st}) \pm \frac{1}{2}\sigma^2_{imp}(K)(t_{fix}-t)}{\sigma_{imp}(K)\sqrt{t_{fix}-t}},$$

Here

(2.19c) $$L(t,\tau_{st}) = \frac{Z(t;\tau_{st}) - Z(t;\tau_{end})}{\beta Z(t;\tau_{end})} + \mathrm{bs}(\tau_{st})$$

is floorlet's forward rate as seen at date t. Today's floorlet value is simply

(2.20a) $$F(0,\tau_{st},K) = \beta D(\tau_{end})\left\{K\mathcal{N}(d_1) - L_0(\tau)\mathcal{N}(d_2)\right\},$$

where

(2.20b) $$d_{1,2} = \frac{\log K/L_0(\tau_{st}) \pm \frac{1}{2}\sigma^2_{imp}(K)t_{fix}}{\sigma_{imp}(K)\sqrt{t_{fix}}},$$

and where today's forward Libor rate is

(2.20c) $$L_0(\tau_{st}) = \frac{D(\tau_{st}) - D(\tau_{end})}{\beta D(\tau_{end})} + \mathrm{bs}(\tau_{st}).$$

To obtain today's price of the accrual swap, note that the effective notional for period j is

(2.21) $$A(0,\tau_{st}) = \frac{\alpha_j R_{fix}}{M_j}\frac{D(t_j)}{D(\tau_{end})}\frac{1}{1+\eta\beta L_0(\tau_{st})}.$$

as seem today. See 2.11b. Putting this together with 2.13a shows that today's price is $V_{cpn}(0) - V_{fund}(0)$, where

(2.22a) $$V_{cpn}(0) = \sum_{j=1}^{n}\frac{\alpha_j R_{fix} D(t_j)}{M_j}\sum_{\tau_{st}=t_{j-1}+1}^{t_j}\frac{\left[1+\eta\beta(R_{\max}-\frac{1}{2}\varepsilon)\right]B_1(\tau_{st}) - \left[1+\eta\beta(R_{\max}+\frac{1}{2}\varepsilon)\right]B_2(\tau_{st})}{\varepsilon\left[1+\eta\beta L_0(\tau_{st})\right]} - \frac{\left[1+\eta\beta(R_{\min}-\frac{1}{2}\varepsilon)\right]B_3(\tau_{st}) - \left[1+\eta\beta(R_{\min}+\frac{1}{2}\varepsilon)\right]B_4(\tau_{st})}{\varepsilon\left[1+\eta\beta L_0(\tau_{st})\right]},$$

(2.22b) $$V_{fund}(0) = D(t_0) - D(t_{\tilde{n}}) + \sum_{i=1}^{\tilde{n}}\mathrm{cvg}(\tilde{t}_{i-1},\tilde{t}_i)(\mathrm{bs}'_i + m_i)D(\tilde{t}_i).$$

Here B_ω are Black's formula at strikes around the boundaries:

(2.22c) $$B_\omega(\tau_{st}) = K_\omega\mathcal{N}(d_1^\omega) - L_0(\tau_{st})\mathcal{N}(d_2^\omega)$$

(2.22d) $$d_{1,2}^\omega = \frac{\log K_\omega/L_0(\tau_{st}) \pm \frac{1}{2}\sigma^2_{imp}(K_\omega)t_{fix}}{\sigma_{imp}(K_\omega)\sqrt{t_{fix}}}$$

with

(2.22e) $$K_{1,2} = R_{\max} \pm \tfrac{1}{2}\varepsilon, \qquad K_{3,4} = R_{\min} \pm \tfrac{1}{2}\varepsilon.$$

Calculating the sum of each day's contribution is very tedious. Normally, one calculates each day's contribution for the current period and two or three months afterward. After that, one usually replaces the sum over dates τ with an integral, and samples the contribution from dates τ one week apart for the next year, and one month apart for subsequent years.

3. Callable accrual swaps. A callable accrual swap is an accrual swap in which the party paying the coupon leg has the right to cancel on any coupon date after a lock-out period expires. For example, a 10NC3 with 5 business days notice can be called on any coupon date, starting on the third anniversary, provided the appropriate notice is given 5 days before the coupon date. We will value the accrual swap from the viewpoint of the receiver, who would price the callable accrual swap as the full accrual swap (coupon leg minus funding leg) *minus* the Bermudan option to enter into the receiver accrual swap. So a 10NC3 cancellable quarterly accrual swap would be priced as the 10 year bullet quarterly receiver accrual swap minus the Bermudan option — with quarterly exercise dates starting in year 3 — to receive the remainder of the coupon leg and pay the remainder of the funding leg. Accordingly, here we price Bermudan options into receiver accrual swaps. Bermudan options on payer accrual swaps can be priced similarly.

There are two key requirements in pricing Bermudan accrual swaps. First, as $R_{\min}$ decreases and $R_{\max}$ increases, the value of the Bermudan accrual swap should reduce to the value of an ordinary Bermudan swaption with strike R_{fix}. Besides the obvious theoretical appeal, meeting this requirement allows one to hedge the callability of the accrual swap by selling an offsetting Bermudan swaption. This criterion requires using the same the interest rate model and calibration method for Bermudan accrual notes as would be used for Bermudan swaptions. Following standard practice, one would calibrate the Bermudan accrual note to the "diagonal swaptions" struck at the accrual note's "effective strikes." For example, a 10NC3 accrual swap which is callable quarterly starting in year 3 would be calibrated to the 3 into 7, the 3.25 into 6.75, ..., the 8.75 into 1.25, and the 9 into 1 swaptions. The strike R^i_{eff} for each of these "reference swaptions" would be chosen so that for swaption i,

$$\frac{\text{value of the fixed leg}}{\text{value of the floating leg}} = \frac{\text{value of all accrual swap coupons } j \geq i}{\text{value of the accrual swap's funding leg } \geq i} \tag{3.1}$$

This usually results in strikes R^i_{eff} that are not too far from the money.

In the preceding section we showed that each coupon of the accrual swap can be written as a combination of vanilla floorlets, and therefore the market value of each coupon is known exactly. The second requirement is that the valuation procedure should reproduce today's market value of each coupon exactly. In fact, if there is a 25% chance of exercising into the accrual swap on or before the j^{th} exercise date, the pricing methodology should yield 25% of the vega risk of the floorlets that make up the j^{th} coupon payment. Effectively this means that the pricing methodology needs to use the correct market volatilities for floorlets struck at $R_{\min}$ and $R_{\max}$. This is a fairly stiff requirement, since we now need to match swaptions struck at R^i_{eff} and floorlets struck at $R_{\min}$ and $R_{\max}$. This is why callable range notes are considered heavily skew depedent products.

3.1. Hull-White model. Meeting these requirements would seem to require using a model that is sophisticated enough to match the floorlet smiles exactly, as well as the diagonal swaption volatilities. Such a model would be complex, calibration would be difficult, and most likely the procedure would yield unstable hedges. An alternative approach is to use a much simpler model to match the diagonal swaption prices, and then use "internal adjusters" to match the floorlet volatilities. Here we follow this approach, using the 1 factor *linear Gauss Markov* (LGM) model with internal adjusters to price Bermudan options on accrual swaps. Specifically, we find explicit formulas for the LGM model's prices of standard floorlets. This enables us to compose the accrual swap "payoffs" (the value recieved at each node in the tree if the Bermudan is exercised) as a linear combination of the vanilla floorlets. With the payoffs known, the Bermudan can be evaluated via a standard rollback. The last step is to note that the LGM model misprices the floorlets that make up the accrual swap coupons, and use internal adjusters to correct this mis-pricing.

Internal adjusters can be used with other models, but the mathematics is more complex.

3.1.1. LGM. The 1 factor LGM model is *exactly* the Hull-White model expressed as an HJM model. The 1 factor LGM model has a single state variable x that determines the entire yield curve at any time t.

This model can be summarized in three equations. The first is the Martingale valuation formula: At any date t and state x, the value of any deal is given by the formula,

$$\frac{V(t,x)}{N(t,x)} = \int p(t,x;T,X)\frac{V(T,X)}{N(T,X)}dX \qquad \text{for any } T > t. \tag{3.2a}$$

Here $p(t,x;T,X)$ is the probability that the state variable is in state X at date T, given that it is in state x at date t. For the LGM model, the transition density is Gaussian

$$p(t,x;T,X) = \frac{1}{\sqrt{2\pi[\zeta(T)-\zeta(t)]}}e^{-(X-x)^2/2[\zeta(T)-\zeta(t)]}, \tag{3.2b}$$

with a variance of $\zeta(T)-\zeta(t)$. The numeraire is

$$N(t,x) = \frac{1}{D(t)}e^{h(t)x+\frac{1}{2}h^2(t)\zeta(t)}, \tag{3.2c}$$

for reasons that will soon become apparent. Without loss of generality, one sets $x = 0$ at $t = 0$, and today's variance is zero: $\zeta(0) = 0$.

The ratio

$$\tilde{V}(t,x) \equiv \frac{V(t,x)}{N(t,x)} \tag{3.3a}$$

is usually called the reduced value of the deal. Since $N(0,0) = 1$, today's value coincides with today's reduced value:

$$V(0,0) = \tilde{V}(0,0) \equiv \frac{V(0,0)}{N(0,0)}. \tag{3.3b}$$

So we only have to work with reduced values to get today's prices.

Define $Z(t,x;T)$ to be the value of a zero coupon bond with maturity T, as seen at t,x. It's value can be found by substituting 1 for $V(T,X)$ in the Martingale valuation formula. This yields

$$\tilde{Z}(t,x;T) \equiv \frac{Z(t,x;T)}{N(t,x)} = D(T)e^{-h(T)x-\frac{1}{2}h^2(T)\zeta(t)}. \tag{3.4a}$$

Since the forward rates are defined through

$$Z(t,x;T) \equiv e^{-\int_t^T f(t,x;T')dT'}, \tag{3.4b}$$

taking $-\frac{\partial}{\partial T}\log Z$ shows that the forward rates are

$$f(t,x;T) = f_0(T) + h'(T)x + h'(T)h(T)\zeta(t). \tag{3.4c}$$

This last equation captures the LGM model in a nutshell. The curves $h(T)$ and $\zeta(t)$ are *model parameters* that need to be set by calibration or by *a priori* reasoning. The above formula shows that at any date t, the forward rate curve is given by today's forward rate curve $f_0(T)$ plus x times a second curve $h'(T)$, where x is a Gaussian random variable with mean zero and variance $\zeta(t)$. Thus $h'(T)$ determines possible shapes of the forward curve and $\zeta(t)$ determines the width of the distribution of forward curves. The last term $h'(T)h(T)\zeta(t)$ is a much smaller convexity correction.

3.1.2. Vanilla prices under LGM. Let $L(t, x; \tau_{st})$ be the forward value of the k month Libor rate for the period τ_{st} to τ_{end}, as seen at t, x. Regardless of model, the forward value of the Libor rate is given by

$$L(t, x; \tau_{st}) = \frac{Z(t, x; \tau_{st}) - Z(t, x; \tau_{end})}{\beta Z(t, x; \tau_{end})} + \mathrm{bs}(\tau_{st}) = L_{true}(t, x; \tau_{st}) + \mathrm{bs}(\tau_{st}), \tag{3.5a}$$

where

$$\beta = \mathrm{cvg}(\tau_{st}, \tau_{end}) \tag{3.5b}$$

is the day count fraction of the interval. Here L_{true} is the forward "true rate" for the interval and $\mathrm{bs}(\tau)$ is the Libor rate's basis spread for the period starting at τ.

Let $F(t, x; \tau_{st}, K)$ be the value at t, x of a floorlet with strike K on the Libor rate $L(t, x; \tau_{st})$. On the fixing date τ_{fix} the payoff is

$$F(\tau_{fix}, x_{fix}; \tau_{st}, K) = \beta \left[K - L(\tau_{fix}, x_{fix}; \tau_{st})\right]^{+} Z(\tau_{fix}, x_{fix}; \tau_{end}), \tag{3.6}$$

where x_{fix} is the state variable on the fixing date. Substituting for $L(\tau_{ex}, x_{ex}; \tau_{st})$, the payoff becomes

$$\frac{F(\tau_{fix}, x_{fix}; \tau_{st}, K)}{N(\tau_{fix}, x_{fix})} = \frac{Z(\tau_{fix}, x_{fix}; \tau_{end})}{N(\tau_{fix}, x_{fix})} \left[1 + \beta(K - \mathrm{bs}(\tau_{st})) - \frac{Z(\tau_{fix}, x_{fix}; \tau_{st})}{Z(\tau_{fix}, x_{fix}; \tau_{end})}\right]^{+}. \tag{3.7a}$$

Knowing the value of the floorlet on the fixing date, we can use the Martingale valuation formula to find the value on any earlier date t:

$$\frac{F(t, x; \tau_{st}, K)}{N(t, x)} = \frac{1}{\sqrt{2\pi[\zeta_{fix} - \zeta]}} \int e^{-(x_{fix} - x)^2/2[\zeta_{fix} - \zeta]} \frac{F(\tau_{fix}, x_{fix}; \tau_{st}, K)}{N(\tau_{fix}, x_{fix})} dx_{fix}, \tag{3.7b}$$

where $\zeta_{fix} = \zeta(\tau_{fix})$ and $\zeta = \zeta(t)$. Substituting the zero coupon bond formula 3.4a and the payoff 3.7a into the integral yields

$$F(t, x; \tau_{st}, K) = Z(t, x; \tau_{end}) \left\{[1 + \beta(K - \mathrm{bs})]\mathcal{N}(\lambda_1) - [1 + \beta(L - \mathrm{bs})]\mathcal{N}(\lambda_2)\right\}, \tag{3.8a}$$

where

$$\lambda_{1,2} = \frac{\log\left(\dfrac{1 + \beta(K - \mathrm{bs})}{1 + \beta(L - \mathrm{bs})}\right) \pm \frac{1}{2}(h_{end} - h_{st})^2 \left[\zeta_{fix} - \zeta(t)\right]}{(h_{end} - h_{st}) \sqrt{\zeta_{fix} - \zeta(t)}}, \tag{3.8b}$$

and where

$$\begin{aligned} L \equiv L(t, x; \tau_{st}) &= \frac{1}{\beta}\left(\frac{Z(t, x; \tau_{st})}{Z(t, x; \tau_{end})} - 1\right) + \mathrm{bs}(\tau_{st}) \\ &= \frac{1}{\beta}\left(\frac{D_{st}}{D_{end}} e^{(h_{end} - h_{st})x + \frac{1}{2}(h_{end}^2 - h_{st}^2)\zeta} - 1\right) + \mathrm{bs}(\tau_{st}) \end{aligned} \tag{3.8c}$$

is the forward Libor rate for the period τ_{st} to τ_{end}, as seen at t, x. Here $h_{st} = h(\tau_{st})$ and $h_{end} = h(\tau_{end})$.

For future reference, it is convenient to split off the zero coupon bond value $Z(t, x; \tau_{end})$. So define the *forwarded floorlet value* by

$$F_f(t, x; \tau_{st}, K) = \frac{F(t, x; \tau_{st}, K)}{Z(t, x; \tau_{end})} \tag{3.9}$$
$$= [1 + \beta(K - \text{bs})]\mathcal{N}(\lambda_1) - [1 + \beta(L(t, x; \tau_{st}) - \text{bs})]\mathcal{N}(\lambda_2).$$

Equations 3.8a and 3.9 are just Black's formulas for the value of a European put option on a log normal asset, provided we identify

$$1 + \beta(L - \text{bs}) = \text{asset's forward value,} \tag{3.10a}$$

$$1 + \beta(K - \text{bs}) = \text{strike,} \tag{3.10b}$$

$$\tau_{end} = \text{settlement date, and} \tag{3.10c}$$

$$(h_{end} - h_{st}) \frac{\sqrt{\zeta_{fix} - \zeta}}{\sqrt{t_{fix} - t}} = \sigma = \text{ asset volatility,} \tag{3.10d}$$

where $t_{fix} - t$ is the time-to-exercise. One should not confuse σ, which is the floorlet's "price volatility," with the commonly quoted "rate volatility."

3.1.3. Rollback. Obtaining the value of the Bermudan is straightforward, given the explicit formulas for the floorlets, . Suppose that the LGM model has been calibrated, so the "model parameters" $h(t)$ and $\zeta(t)$ are known. (In Appendix A we show one popular calibration method). Let the Bermudan's notification dates be $t^{ex}_{k_0}, t^{ex}_{k_0+1}, \ldots, t^{ex}_K$. Suppose that if we exercise on date t^{ex}_k, we receive all coupon payments for the intervals $k + 1, \ldots, n$ and recieve all funding leg payments for intervals $i_k, i_k + 1, \ldots, \tilde{n}$.

The rollback works by induction. Assume that in the previous rollback steps, we have calculated the reduced value

$$\frac{V^+(t^{ex}_k, x)}{N(t^{ex}_k, x)} = \text{value at } t^{ex}_k \text{ of all remaining exercises } t^{ex}_{k+1}, t^{ex}_{k+2} \ldots, t^{ex}_K \tag{3.11a}$$

at each x. We show how to take one more step backwards, finding the value which includes the exercise t^{ex}_k at the preceding exercise date:

$$\frac{V^+(t^{ex}_{k-1}, x)}{N(t^{ex}_{k-1}, x)} = \text{value at } t^{ex}_{k-1} \text{ of all remaining exercises } t^{ex}_k, t^{ex}_{k+1}, t^{ex}_{k+2} \ldots, t^{ex}_K. \tag{3.11b}$$

Let $P_k(x)/N(t^{ex}_k, x)$ be the (reduced) value of the payoff obtained if the Bermudan is exercised at t^{ex}_k. As seen at the exercise date t^{ex}_k the effective notional for date τ_{st} is

$$A(t^{ex}_k, x, \tau_{st}) = \frac{\alpha_j R_{fix}}{M_j} \frac{Z(t^{ex}_k, x; t_j)}{Z(t^{ex}_k, x; \tau_{end})} \frac{1}{1 + \eta\beta L_f(t^{ex}_k, x; \tau_{st})}, \tag{3.12a}$$

where we recall that

$$\eta = \frac{\tau_{end}(\tau_{st}) - t_j}{\tau_{end}(\tau_{st}) - \tau_{st}}, \qquad \beta = \text{cvg}(\tau_{st}, \tau_{end}(\tau_{st})). \tag{3.12b}$$

Reconstructing the reduced value of the payoff (see equation 2.15) yields

$$
\begin{aligned}
(3.13a)\qquad & \frac{P_k(x)}{N(t_k^{ex},x)} = \\
& \sum_{j=k+1}^{n} \frac{\alpha_j R_{fix} Z(t_k^{ex},x;t_j)}{\varepsilon\beta M_j N(t_k^{ex},x)} \sum_{\tau_{st}=t_{j-1}+1}^{t_j} \left\{ \frac{1+\eta\beta(R_{\max}-\frac{1}{2}\varepsilon)}{1+\eta\beta L_f(t_k^{ex},x;\tau_{st})} F_f(t_k^{ex},x;\tau_{st},R_{\max}+\tfrac{1}{2}\varepsilon) \right. \\
& \qquad - \frac{1+\eta\beta(R_{\max}+\frac{1}{2}\varepsilon)}{1+\eta\beta L_f(t_k^{ex},x;\tau_{st})} F_f(t_k^{ex},x;\tau_{st},R_{\max}-\tfrac{1}{2}\varepsilon) \\
& \qquad - \frac{1+\eta\beta(R_{\min}-\frac{1}{2}\varepsilon)}{1+\eta\beta L_f(t_k^{ex},x;\tau_{st})} F_f(t_k^{ex},x;\tau_{st},R_{\min}+\tfrac{1}{2}\varepsilon) \\
& \qquad \left. + \frac{1+\eta\beta(R_{\min}+\frac{1}{2}\varepsilon)}{1+\eta\beta L_f(t_k^{ex},x;\tau_{st})} F_f(t_k^{ex},x;\tau_{st},R_{\min}-\tfrac{1}{2}\varepsilon) \right\} \\
& - \frac{Z(t_k^{ex},x,\tilde{t}_{i_k-1}) - Z(t_k^{ex},x,\tilde{t}_{\tilde{n}})}{N(t_k^{ex},x)} - \sum_{i=i_k+1}^{\tilde{n}} \mathrm{cvg}(\tilde{t}_{i-1},\tilde{t}_i)(\mathrm{bs}_i + m_i) \frac{Z(t_k^{ex},x,\tilde{t}_i)}{N(t_k^{ex},x)}.
\end{aligned}
$$

This payoff includes only zero coupon bonds and floorlets, so we can calculate this reduced payoff explicitly using the previously derived formula 3.9. The reduced valued *including* the k^{th} exercise is clearly

$$
(3.13b)\qquad \frac{V(t_k^{ex},x)}{N(t_k^{ex},x)} = \max\left\{ \frac{P_k(x)}{N(t_k^{ex},x)}, \frac{V^+(t_k^{ex},x)}{N(t_k^{ex},x)} \right\} \qquad \text{at each } x.
$$

Using the Martingale valuation formula we can "roll back" to the preceding exercise date by using finite differences, trees, convolution, or direct integration to compute the integral

$$
(3.13c)\qquad \frac{V^+(t_{k-1}^{ex},x)}{N(t_{k-1}^{ex},x)} = \frac{1}{\sqrt{2\pi[\zeta_k-\zeta_{k-1}]}} \int \frac{V(t_k^{ex},X)}{N(t_k^{ex},X)} e^{-(X-x)^2/2[\zeta_k-\zeta_{k-1}]} dX
$$

at each x. Here $\zeta_k = \zeta(t_k^{ex})$ and $\zeta_{k-1} = \zeta(t_{k-1}^{ex})$.

At this point we have moved from t_k^{ex} to the preceding exercise date t_{k-1}^{ex}. We now repeat the procedure: at each x we take the max of $V^+(t_{k-1}^{ex},x)/N(t_{k-1}^{ex},x)$ and the payoff $P_{k-1}(x)/N(t_{k-1}^{ex},x)$ for t_{k-1}^{ex}, and then use the valuation formula to roll-back to the preceding exercise date t_{k-2}^{ex}, etc. Eventually we work our way througn the first exercise $V(t_{k_0}^{ex},x)$. Then today's value is found by a final integration:

$$
(3.14)\qquad V(0,0) = \frac{V(0,0)}{N(0,0)} = \frac{1}{\sqrt{2\pi\zeta_{k_0}}} \int \frac{V(t_{k_0}^{ex},X)}{N(t_{k_0}^{ex},X)} e^{-X^2/2\zeta_{k_0}} dX.
$$

3.2. Using internal adjusters. The above pricing methodology satisfies the first criterion: Provided we use LGM (Hull-White) to price our Bermudan swaptions, and provided we use the same calibration method for accrual swaps as for Bermudan swaptions, the above procedure will yield prices that reduce to the Bermudan prices as $R_{\min}$ goes to zero and $R_{\max}$ becomes large.

However the LGM model yields the following formulas for today's values of the standard floorlets:

$$
(3.15a)\qquad F(0,0;\tau_{st},K) = D(\tau_{end})\left\{[1+\beta(K-\mathrm{bs})]\mathcal{N}(\lambda_1) - [1+\beta(L_0-\mathrm{bs})]\mathcal{N}(\lambda_2)\right\}
$$

where

$$
(3.15b)\qquad \lambda_{1,2} = \frac{\log\left(\dfrac{1+\beta(K-\mathrm{bs})}{1+\beta(L_0-\mathrm{bs})}\right) \pm \frac{1}{2}\sigma_{\mathrm{mod}}^2 t_{fix}}{\sigma_{\mathrm{mod}}\sqrt{t_{fix}}}.
$$

Here

$$L_0 = \frac{D_{st} - D_{end}}{\beta D_{end}} + \mathrm{bs}(\tau_{st}) \tag{3.15c}$$

is today's forward value for the Libor rate, and

$$\sigma_{\mathrm{mod}} = (h_{end} - h_{st})\sqrt{\zeta_{fix}/t_{fix}} \tag{3.15d}$$

is the asset's log normal volatility according to the LGM model. We did not calibrate the LGM model to these floorlets. It is virtually certain that matching today's market prices for the floorlets will require using an implied (price) volatility σ_{mkt} which differs from $\sigma_{\mathrm{mod}} = (h_{end} - h_{st})\sqrt{\zeta_{fix}/t_{fix}}$.

3.2.1. Obtaining the market vol. Floorlets are quoted in terms of the ordinary (rate) vol. Suppose the rate vol is quoted as $\sigma_{imp}(K)$. Then today's market price of the floorlet is

$$F_{mkt}(\tau_{st}, K) = \beta D(\tau_{end}) \left\{ K\mathcal{N}(d_1) - L_0\mathcal{N}(d_2) \right\} \tag{3.16a}$$

where

$$d_{1,2} = \frac{\log K/L_0 \pm \frac{1}{2}\sigma_{imp}^2(K)t_{fix}}{\sigma_{imp}(K)\sqrt{t_{fix}}} \tag{3.16b}$$

The price vol σ_{mkt} is the volatility that equates the LGM floorlet value to this market value. It is defined implicitly by

$$[1 + \beta(K - \mathrm{bs})]\mathcal{N}(\lambda_1) - [1 + \beta(L_0 - \mathrm{bs})]\mathcal{N}(\lambda_2) = \beta K\mathcal{N}(d_1) - \beta L_0\mathcal{N}(d_2), \tag{3.17a}$$

with

$$\lambda_{1,2} = \frac{\log\left(\dfrac{1 + \beta(K - \mathrm{bs})}{1 + \beta(L_0 - \mathrm{bs})}\right) \pm \frac{1}{2}\sigma_{mkt}^2 t_{fix}}{\sigma_{mkt}\sqrt{t_{fix}}} \tag{3.17b}$$

$$d_{1,2} = \frac{\log K/L_0 \pm \frac{1}{2}\sigma_{imp}^2(K)t_{fix}}{\sigma_{imp}(K)\sqrt{t_{fix}}} \tag{3.17c}$$

Equivalent vol techniques can be used to find the price vol $\sigma_{mkt}(K)$ which corresponds to the market-quoted implied rate vol $\sigma_{imp}(K)$:

$$\frac{\sigma_{imp}(K)}{1 + \frac{1}{24}\sigma_{imp}^2 t_{fix} + \frac{1}{5760}\sigma_{imp}^4 t_{fix}^2 + \cdots} = \frac{\sigma_{mkt}(K)}{1 + \frac{1}{24}\sigma_{mkt}^2 t_{fix} + \frac{1}{5760}\sigma_{mkt}^4 t_{fix}^2} \frac{\log L_0/K}{\log\left(\dfrac{1 + \beta(L_0 - \mathrm{bs})}{1 + \beta(K - \mathrm{bs})}\right)} \tag{3.18}$$

If this approximation is not sufficiently accurate, we can use a single Newton step to attain any reasonable accuracy.

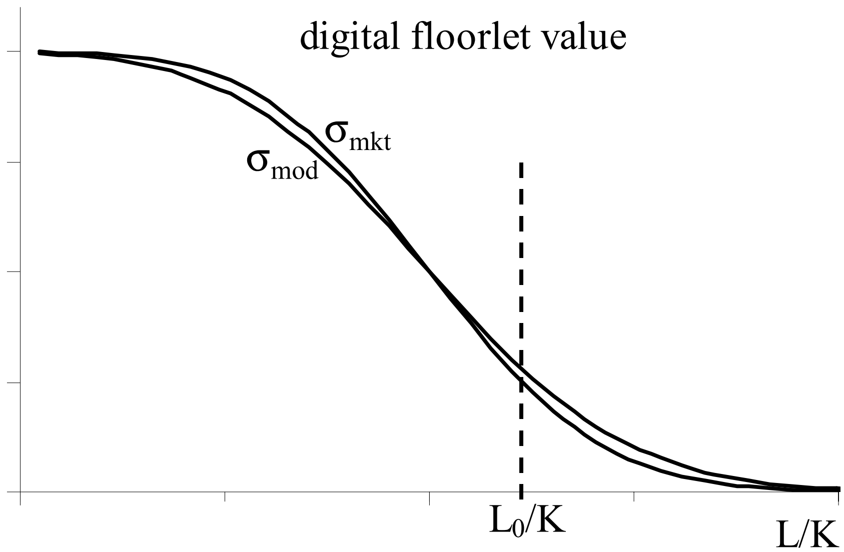


FIG. 3.1. *Unadjusted and adjusted digital payoff*

3.2.2. Adjusting the price vol. The price vol σ_{mkt} obtained from the market price will not match the LGM model's price vol $\sigma_{\text{mod}} = (h_{end} - h_{st})\sqrt{\zeta_{fix}/t_{fix}}$. This is easily remedied using an internal adjuster. All one does is multiply the model volatility with the factor needed to bring it into line with the actual market volatility, and use this factor when calculating the payoffs. Specifically, in calculating each payoff $P_k(x)/N(t_k^{ex}, x)$ in the rollback (see eq. 3.13a), one makes the replacement

$$(h_{end} - h_{st})\sqrt{\zeta_{fix} - \zeta(t_k^{ex})} \Longrightarrow (h_{end} - h_{st})\sqrt{\zeta_{fix} - \zeta(t)}\frac{\sigma_{mkt}}{\sigma_{\text{mod}}} \tag{3.19}$$

$$= \sqrt{1 - \zeta(t_k^{ex})/\zeta(t_{fix})}\sigma_{mkt}\sqrt{t_{fix}}. \tag{3.20}$$

With the internal adjusters, the pricing methodology now satisfies the second criteria: it agrees with all the vanilla prices that make up the range note coupons. Essentially, all the adjuster does is to slightly "sharpen up" or "smear out" the digital floorlet's payoff to match today's value at L_0/K. This results in slightly positive or negative price corrections at various values of L/K, but these corrections average out to zero when averaged over all L/K. Making this volatility adjustment is vastly superior to the other commonly used adjustment method, which is to add in a fictitious "exercise fee" to match today's coupon value. Adding a fee gives a positive or negative bias to the payoff for all L/K, even far from the money, where the payoff was certain to have been correct.

Meeting the second criterion forced us to go outside the model. It is possible that there is a subtle arbitrage to our pricing methodology. (There may or may not be an arbitrage free model in which extra factors — positively or negatively correlated with x — enable us to obtain exactly these floorlet prices while leaving our Gaussian rollback unaffected). However, not matching today's price of the underlying accrual swap would be a direct and immediate arbitrage.

4. Range notes and callable range notes. In an accrual swap, the coupon leg is exchanged for a funding leg, which is normally a standard Libor leg plus a margin. Unlike a bond, there is no principle at risk. The only credit risk is for the difference in value between the coupon leg and the floating leg payments; even this difference is usually collateralized through various inter-dealer arrangements. Since swaps are indivisible, liquidity is not an issue: they can be unwound by transferring a payment of the accrual swap's mark-to-market value. For these reasons, there is no detectable OAS in pricing accrual swaps.

A range note is an actual bond which pays the coupon leg on top of the principle repayments; there is no funding leg. For these deals, the issuer's credit-worthiness is a key concern. One needs to use an *option adjusted spread* (OAS) to obtain the extra discounting reflecting the counterparty's credit spread and liquidity. Here we analyze bullet range notes, both uncallable and callable.

The coupons C_j of these notes are set by the number of days an index (usually Libor) sets in a specified range, just like accrual swaps:

$$C_j = \sum_{\tau_{st}=t_{j-1}+1}^{t_j} \frac{\alpha_j R_{fix}}{M_j} \begin{cases} 1 & \text{if } R_{\min} \le L(\tau_{st}) \le R_{\max} \\ 0 & \text{otherwise} \end{cases}, \tag{4.1a}$$

where $L(\tau_{st})$ is k month Libor for the interval τ_{st} to $\tau_{end}(\tau_{st})$, and where α_j and M_j are the day count fraction and the total number of days in the j^{th} coupon interval t_{j-1} to t_j. In addition, these range notes repay the principle on the final pay date, so the (bullet) range note payments are:

$$C_j \qquad \text{paid on } t_j, \qquad j = 1, 2, \dots n-1, \tag{4.1b}$$

$$1 + C_n \qquad \text{paid on } t_n. \tag{4.1c}$$

For callable range notes, let the notification on dates be t_k^{ex} for $k = k_0, k_0+1, \dots, K-1, K$ with $K < n$. Assume that if the range note is called on t_k^{ex}, then the strike price K_k is paid on coupon date t_k and the payments C_j are cancelled for $j = k+1, \dots, n$.

4.1. Modeling option adjusted spreads. Suppose a range note is issued by issuer A. Define $Z_A(t, x; T)$ to be the value of a dollar paid by the note on date T, as seen at t, x. We assume that

$$Z_A(t, x; T) = Z(t, x; T) \frac{\Lambda(T)}{\Lambda(t)}, \tag{4.2}$$

where $Z(t, x; T)$ is the value according to the Libor curve, and

$$\Lambda(\tau) = \frac{D_A(\tau)}{D(\tau)} e^{-\gamma\tau}. \tag{4.3}$$

Here γ is the OAS of the range note. The choice of the discount curve $D_A(\tau)$ depends on what we wish the OAS to measure. If one wishes to find the range note's value relative to the issuer's other bonds, then one should use the issuer's discount curve for $D_A(\tau)$; the OAS then measures the note's richness or cheapness compared to the other bonds of issuer A. If one wishes to find the note's value relative to its credit risk, then the OAS calculation should use the issuer's "risky discount curve" or for the issuer's credit rating's risky discount curve for $D_A(\tau)$. If one wishes to find the absolute OAS, then one should use the swap market's discount curve $D(\tau)$, so that $\Lambda(\tau)$ is just $e^{-\gamma\tau}$.

When valuing a non-callable range note, we are just determining which OAS γ is needed to match the current price. I.e., the OAS needed to match the market's idiosyncratic preference or adversion of the bond. When valuing a callable range note, we are making a much more powerful assumption. By assuming that the same γ can be used in evaluating the calls, we are assuming that
(1) the issuer would re-issue the bonds if it could do so more cheaply, and
(2) on each exercise date in the future, the issuer could issue debt at the same OAS that prevails on today's bond.

4.2. Non-callable range notes. Range note coupons are fixed by Libor settings and other issuer-independent criteria. Thus the value of a range note is obtained by leaving the coupon calculations alone, and replacing the coupon's discount factors $D(t_j)$ with the bond-appropriate $D_A(t_j)e^{-\gamma t_j}$:

(4.4a) $V_A(0) =$

$$\sum_{j=1}^{n} \frac{\alpha_j R_{fix} D_A(t_j) e^{-\gamma t_j}}{M_j} \sum_{\tau_{st}=t_{j-1}+1}^{t_j} \frac{\left[1+\eta\beta(R_{\max}-\frac{1}{2}\varepsilon)\right] B_1(\tau_{st}) - \left[1+\eta\beta(R_{\max}+\frac{1}{2}\varepsilon)\right] B_2(\tau_{st})}{\varepsilon\left[1+\eta\beta L_0(\tau_{st})\right]}$$
$$- \frac{\left[1+\eta\beta(R_{\min}-\frac{1}{2}\varepsilon)\right] B_3(\tau_{st}) - \left[1+\eta\beta(R_{\min}+\frac{1}{2}\varepsilon)\right] B_4(\tau_{st})}{\varepsilon\left[1+\eta\beta L_0(\tau_{st})\right]}$$
$$+ D_A(t_n)e^{-\gamma t_n}.$$

Here the last term $D_A(t_n)e^{-\gamma t_n}$ is the value of the notional repaid at maturity. As before, the B_ω are Black's formulas,

(4.4b)
$$B_\omega(\tau_{st}) = K_j \mathcal{N}(d_1^\omega) - L_0(\tau_{st})\mathcal{N}(d_2^\omega)$$

(4.4c)
$$d_{1,2}^\omega = \frac{\log K_\omega / L_0(\tau_{st}) \pm \frac{1}{2}\sigma_{imp}^2(K_\omega)t_{fix}}{\sigma_{imp}(K_\omega)\sqrt{t_{fix}}}$$

(4.4d)
$$K_{1,2} = R_{\max} \pm \tfrac{1}{2}\varepsilon, \qquad K_{3,4} = R_{\min} \pm \tfrac{1}{2}\varepsilon,$$

and $L_0(\tau)$ is today's forward rate:

(4.4e)
$$L_0(\tau_{st}) = \frac{D(\tau_{st}) - D(\tau_{end})}{\beta D(\tau_{end})}$$

Finally,

(4.4f)
$$\eta = \frac{\tau_{end} - t_j}{\tau_{end} - \tau_{st}}.$$

4.3. Callable range notes. We price the callable range notes via the same Hull-White model as used to price the cancelable accrual swap. We just need to adjust the coupon discounting in the payoff function. Clearly the value of the callable range note is the value of the non-callable range note minus the value of the call:

(4.5)
$$V_A^{callable}(0) = V_A^{bullet}(0) - V_A^{Berm}(0).$$

Here $V_A^{bullet}(0)$ is the today's value of the non-callable range note in 4.4a, and $V_A^{Berm}(0)$ is today's value of the Bermudan option. This Bermudan option is valued using exactly the same rollback procedure as before, except that now the payoff is

$$\frac{P_k(x)}{N(t_k^{ex},x)} = \tag{4.6a}$$

$$\sum_{j=k+1}^{n} \frac{\alpha_j R_{fix} Z_A(t_k^{ex},x;t_j)}{\varepsilon\beta M_j N(t_k^{ex},x)} \sum_{\tau_{st}=t_{j-1}+1}^{t_j} \left\{ \frac{1+\eta\beta(R_{\max}-\frac{1}{2}\varepsilon)}{1+\eta\beta L_f(t_k^{ex},x;\tau_{st})} F_f(t_k^{ex},x;\tau_{st},R_{\max}+\tfrac{1}{2}\varepsilon) \right.$$
$$- \frac{1+\eta\beta(R_{\max}+\frac{1}{2}\varepsilon)}{1+\eta\beta L_f(t_k^{ex},x;\tau_{st})} F_f(t_k^{ex},x;\tau_{st},R_{\max}-\tfrac{1}{2}\varepsilon)$$
$$- \frac{1+\eta\beta(R_{\min}-\frac{1}{2}\varepsilon)}{1+\eta\beta L_f(t_k^{ex},x;\tau_{st})} F_f(t_k^{ex},x;\tau_{st},R_{\min}+\tfrac{1}{2}\varepsilon)$$
$$\left. + \frac{1+\eta\beta(R_{\min}+\frac{1}{2}\varepsilon)}{1+\eta\beta L_f(t_k^{ex},x;\tau_{st})} F_f(t_k^{ex},x;\tau_{st},R_{\min}-\tfrac{1}{2}\varepsilon) \right\}$$
$$+ \frac{Z_A(t_k^{ex},x,t_n)}{N(t_k^{ex},x)} - K_k \frac{Z_A(t_k^{ex},x,t_k)}{N(t_k^{ex},x)} \tag{4.6b}$$

Here the *bond specific* reduced zero coupon bond value is

$$\frac{Z_A(t_k^{ex},x,T)}{N(t_k^{ex},x)} = \frac{D(t_k^{ex})}{D_A(t_k^{ex})} D_A(T) e^{-\gamma(T-t_k^{ex})} e^{-h(T)x-\frac{1}{2}h^2(T)\zeta_k^{ex}}, \tag{4.6c}$$

the (adjusted) forwarded floorlet value is

$$F_f(t_k^{ex},x;\tau_{st},K) = [1+\beta(K-\mathrm{bs})]\mathcal{N}(\lambda_1) - [1+\beta(L(t_k^{ex},x;\tau_{st})-\mathrm{bs})]\mathcal{N}(\lambda_2)$$

$$\lambda_{1,2} = \frac{\log\left(\dfrac{1+\beta(K-\mathrm{bs})}{1+\beta(L-\mathrm{bs})}\right) \pm \frac{1}{2}[1-\zeta(t_k^{ex})/\zeta(t_{fix})]\sigma_{mkt}^2 t_{fix}}{\sqrt{1-\zeta(t_k^{ex})/\zeta(t_{fix})}\,\sigma_{mkt}\sqrt{t_{fix}}}, \tag{4.6d}$$

and the forward Libor value is

$$L \equiv L\left(t_k^{ex},x;\tau_{st}\right) = \frac{1}{\beta}\left(\frac{Z(t_k^{ex},x;\tau_{st})}{Z(t_k^{ex},x;\tau_{end})} - 1\right) + \mathrm{bs}(\tau_{st}) \tag{4.6e}$$

$$= \frac{1}{\beta}\left(\frac{D_{st}}{D_{end}} e^{(h_{end}-h_{st})x-\frac{1}{2}(h_{end}^2-h_{st}^2)\zeta_k^{ex}} - 1\right) + \mathrm{bs}(\tau_{st}) \tag{4.6f}$$

The only remaining issue is calibration. For range notes, we should use constant mean reversion and calibrate along the diagonal, exactly as we did for the cancelable accrual swaps. We only need to specify the strikes of the reference swaptions. A good method is to transfer the basis spreads and margin to the coupon leg, and then match the ratio of the coupon leg to the floating leg. For exercise on date t_k, this ratio yields

$$\lambda_k = \tag{4.7a}$$

$$\sum_{j=k+1}^{n} \frac{\alpha_j R_{fix} D_A(t_j) e^{-\gamma(t_j-t_k)}}{M_j K_k D_A(t_k)} \sum_{\tau_{st}=t_{j-1}+1}^{t_j} \left\{ \frac{\left[1+\eta\beta(R_{\max}-\frac{1}{2}\varepsilon)\right] B_1(\tau_{st}) - \left[1+\eta\beta(R_{\max}+\frac{1}{2}\varepsilon)\right] B_1(\tau_{st})}{\varepsilon\left[1+\eta\beta L_0(\tau_{st})\right]} \right.$$
$$\left. - \frac{\left[1+\eta\beta(R_{\min}-\frac{1}{2}\varepsilon)\right] B_3(\tau_{st}) - \left[1+\eta\beta(R_{\min}+\frac{1}{2}\varepsilon)\right] B_3(\tau_{st})}{1+\eta\beta L_f(t_k^{ex},x;\tau_{st})} \right\}$$
$$+ \frac{D_A(t_n) e^{-\gamma(t_n-t_k)}}{K_k D_A(t_k)}$$

As before, the B_j are dimensionless Black formulas,

$$B_\omega(\tau_{st}) = K_\omega \mathcal{N}(d_1^\omega) - L_0(\tau_{st})\mathcal{N}(d_2^\omega) \tag{4.7b}$$

$$d_{1,2}^\omega = \frac{\log K_\omega / L_0(\tau_{st}) \pm \frac{1}{2}\sigma_{imp}^2(K_\omega) t_{fix}}{\sigma_{imp}(K_\omega)\sqrt{t_{fix}}} \tag{4.7c}$$

$$K_{1,2} = R_{\max} \pm \tfrac{1}{2}\varepsilon, \qquad K_{3,4} = R_{\min} \pm \tfrac{1}{2}\varepsilon, \tag{4.7d}$$

and $L_0(\tau_{st})$ is today's forward rate:

Appendix A. Calibrating the LGM model.

The are several methods of calibrating the LGM model for pricing a Bermudan swaption. The most popular method is to choose a constant mean reversion κ, and then calibrate on the diagonal European swaptions making up the Bermudan. In the LGM model, a "constant mean reversion κ" means that the model function $h(t)$ is given by

$$h(t) = \frac{1 - e^{-\kappa t}}{\kappa}. \tag{A.1}$$

Usually the value of κ is selected from a table of values that are known to yield the correct market prices of liquid Bermudans; It is known empirically that the needed mean reversion parameters are very, very stable, changing little from year to year.

κ	1Y	2Y	3Y	4Y	5Y	7Y	10Y
1M	-1.00%	-0.50%	-0.25%	-0.25%	-0.25%	-0.25%	-0.25%
3M	-0.75%	-0.25%	0.00%	0.00%	0.00%	0.00%	0.00%
6M	-0.50%	0.00%	0.25%	0.25%	0.25%	0.25%	0.25%
1Y	0.00%	0.25%	0.50%	0.50%	0.50%	0.50%	0.50%
3Y	0.25%	0.50%	1.00%	1.00%	1.00%	1.00%	1.00%
5Y	0.50%	1.00%	1.25%	1.25%	1.25%	1.25%	1.25%
7Y	1.00%	1.25%	1.50%	1.50%	1.50%	1.50%	1.50%
10Y	1.50%	1.50%	1.75%	1.75%	1.75%	1.75%	1.75%

Table A.1

Mean reverssion κ for Bermudan swaptions. Rows are time-to-first exercise; columns are tenor of the longest underlying swap obtained upon exercise.

With $h(t)$ known, we only need determine $\zeta(t)$ by calibrating to European swaptions. Consider a European swaption with notification date t_{ex}. Suppose that if one exercises the option, one recieves a fixed leg worth

$$V_{fix}(t, x) = \sum_{i=1}^{n} R_{fix}\, \mathrm{cvg}(t_{i-1}, t_i, \mathrm{dcb}_{fix}) Z(t, x; t_i), \tag{A.2a}$$

and pays a floating leg worth

$$V_{flt}(t, x) = Z(t, x; t_0) - Z(t, x; t_n) + \sum_{i=1}^{n} \mathrm{cvg}(t_{i-1}, t_i, \mathrm{dcb}_{flt})\, \mathrm{bs}_i\, Z(t, x; t_i). \tag{A.2b}$$

Here $\text{cvg}(t_{i-1}, t_i, \text{dcb}_{fix})$ and $\text{cvg}(t_{i-1}, t_i, \text{dcb}_{flt})$ are the day count fractions for interval i using the fixed leg and floating leg day count bases. (For simplicity, we are cheating slightly by applying the floating leg's basis spread at the frequency of the fixed leg. Mea culpa). Adjusting the basis spread for the difference in the day count bases

$$\text{bs}_i^{new} = \frac{\text{cvg}(t_{i-1}, t_i, \text{dcb}_{flt})}{\text{cvg}(t_{i-1}, t_i, \text{dcb}_{fix})} \text{bs}_i \tag{A.3}$$

allows us to write the value of the swap as

$$\begin{aligned} V_{swap}(t,x) &= V_{fix}(t,x) - V_{flt}(t,x) \\ &= \sum_{i=1}^{n} (R_{fix} - \text{bs}_i) \, \text{cvg}(t_{i-1}, t_i, \text{dcb}_{fix}) Z(t,x;t_i) + Z(t,x;t_n) - Z(t,x;t_0) \end{aligned} \tag{A.4}$$

Under the LGM model, today's value of the swaption is

$$V_{swptn}(0,0) = \frac{1}{\sqrt{2\pi\zeta(t_{ex})}} \int e^{-x_{ex}^2/2\zeta(t_{ex})} \frac{[V_{swap}(t_{ex}, x_{ex})]^+}{N(t_{ex}, x_{ex})} dx_{ex} \tag{A.5}$$

Substituting the explicit formulas for the zero coupon bonds and working out the integral yields

$$\begin{aligned} V_{swptn}(0,0) = \sum_{i=1}^{n} (R_{fix} - \text{bs}_i) \, \text{cvg}(t_{i-1}, t_i, \text{dcb}_{fix}) D(t_i) \mathcal{N}\left(\frac{y + [h(t_i) - h(t_0)]\, \zeta_{ex}}{\sqrt{\zeta_{ex}}} \right) \\ + D(t_n) \mathcal{N}\left(\frac{y + [h(t_n) - h(t_0)]\, \zeta_{ex}}{\sqrt{\zeta_{ex}}} \right) - D(t_0) \mathcal{N}\left(\frac{y}{\sqrt{\zeta_{ex}}} \right), \end{aligned} \tag{A.6a}$$

where y is determined implicitly via

$$\begin{aligned} \sum_{i=1}^{n} (R_{fix} - \text{bs}_i) \, \text{cvg}(t_{i-1}, t_i, \text{dcb}_{fix}) e^{-[h(t_i)-h(t_0)]y - \frac{1}{2}[h(t_i)-h(t_0)]^2 \zeta_{ex}} \\ + D(t_n) e^{-[h(t_n)-h(t_0)]y - \frac{1}{2}[h(t_n)-h(t_0)]^2 \zeta_{ex}} = D(t_0). \end{aligned} \tag{A.6b}$$

The values of $h(t)$ are known for all t, so the only unknown parameter in this price is $\zeta(t_{ex})$. One can show that the value of the swaption is an increasing function of $\zeta(t_{ex})$, so there is exactly one $\zeta(t_{ex})$ which matches the LGM value of the swaption to its market price. This solution is easily found via a global Newton iteration.

To price a Bermudan swaption, one typcially calibrates on the component Europeans. For, say, a 10NC3 Bermudan swaption struck at 8.2% and callable quarterly, one would calibrate to the 3 into 7 swaption struck at 8.2%, the 3.25 into 6.75 swaption struck at 8.2%, ..., then 8.75 into 1.25 swaption struck at 8.25%, and finally the 9 into 1 swaption struck at 8.2%. Calibrating each swaption gives the value of $\zeta(t)$ on the swaption's exercise date. One generally uses piecewise linear interpolation to obtain $\zeta(t)$ at dates between the exercise dates.

The remaining problem is to pick the strike of the reference swaptions. A good method is to transfer the basis spreads and margin to the coupon leg, and then match the ratio of the coupon leg to the funding leg to the equivalent ratio for a swaption. For the exercise on date t_k, this ratio is defined to be

$$\text{(A.7a)}\quad \lambda_k = \sum_{j=k+1}^{n} \frac{\alpha_j D(t_j)}{M_j D(t_k)} \sum_{\tau_{st}=t_{j-1}+1}^{t_j} \frac{\left[1+\eta\beta(R_{\max}-\frac{1}{2}\varepsilon)\right] B_1(\tau_{st}) - \left[1+\eta\beta(R_{\max}+\frac{1}{2}\varepsilon)\right] B_2(\tau_{st})}{\varepsilon\left[1+\eta\beta L_0(\tau_{st})\right]}$$

$$- \frac{\left[1+\eta\beta(R_{\min}-\frac{1}{2}\varepsilon)\right] B_3(\tau_{st}) - \left[1+\eta\beta(R_{\min}+\frac{1}{2}\varepsilon)\right] B_4(\tau_{st})}{\varepsilon\left[1+\eta\beta L_0(\tau_{st})\right]}$$

$$+ \frac{D(t_n)}{D(t_k)} - \sum_{i=1}^{n} \mathrm{cvg}(t_{i-1}, t_i)(\mathrm{bs}'_i + m_i) \frac{D(t_i)}{D(t_k)}$$

where B_ω are Black's formula at strikes around the boundaries:

$$\text{(A.7b)}\qquad B_\omega(\tau_{st}) = \beta D(\tau_{end}) \left\{K_\omega \mathcal{N}(d_1^\omega) - L_0(\tau_{st})\mathcal{N}(d_2^\omega)\right\}$$

$$\text{(A.7c)}\qquad d_{1,2}^\omega = \frac{\log K_\omega / L_0(\tau_{st}) \pm \frac{1}{2}\sigma_{imp}^2(K_\omega) t_{fix}}{\sigma_{imp}(K_\omega)\sqrt{t_{fix}}}$$

with

$$\text{(A.7d)}\qquad K_{1,2} = R_{\max} \pm \tfrac{1}{2}\varepsilon, \qquad K_{3,4} = R_{\min} \pm \tfrac{1}{2}\varepsilon.$$

This is to be matched to the swaption whose swap starts on t_k and ends on t_n, with the strike R_{fix} chosen so that the equivalent ratio matches the λ_k defined above:

$$\text{(A.7e)}\qquad \lambda_k = \sum_{i=k+1}^{n} (R_{fix} - \mathrm{bs}_i)\, \mathrm{cvg}(t_{i-1}, t_i, \mathrm{dcb}_{fix}) \frac{D(t_i)}{D(t_k)} + \frac{D(t_n)}{D(t_k)}$$

The above methodology works well for deals that are similar to bullet swaptions. For some exotics, such as amortizing deals or zero coupon callables, one may wish to choose both the tenor of the and the strike of the reference swaptions. This allows one to match the exotic deal's duration as well as its moneyness.

Appendix B. Floating rate accrual notes.